Abductive Question-Answer System for the Minimal Logic of Formal Inconsistency

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Outline

Abductive reasoning

1. A knowledge base Γ;
   a phenomenon ϕ, which is unattainable from Γ.

1. A knowledge base Γ;
   a phenomenon ϕ, which is unattainable from Γ.
2. H — an abductive hypothesis;
   ϕ can be computed/derived from Γ' which is equal to Γ augmented with H.

Abductive Question-Answer System

Erotetic Decomposition Principle [?, p. 103]

Transform a principal question into auxiliary questions in such a way that: (i) consecutive auxiliary questions are dependent upon previous questions and, possibly, answers to previous auxiliary questions, and (ii) once auxiliary questions are resolved, the principal question is resolved as well.

AQAS components

Abductive Question-Answer System

Rules for questions procesing

$$\frac{Question}{Question}$$

Rules for answering questions

$$\frac{Question}{Answer}$$

Formulas of $\mathcal{L}^?_{mbC}$

Well-formed formulas of $\mathcal{L}_{mbC}$ ($\mathrm{FOR_{mbC}}$)

$$A, B :: p_i \mid \neg A \mid \ \sim A \mid \ \circ A \mid A \wedge B \mid A \vee B \mid A \rightarrow B$$

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Well-formed formulas of $\mathcal{L}_{mbC}$ ($\mathrm{FOR}_{\mathrm{mbC}}$)

$$A, B :: p_i \mid \neg A \mid \sim A \mid \circ A \mid A \wedge B \mid A \vee B \mid A \rightarrow B$$

Well-formed formulas of $\mathcal{L}^{+}_{mbC}$ ($\mathrm{FOR}^{+}_{\mathrm{mbC}}$)

$$\mathrm{FOR}^{+}_{\mathrm{mbC}} = \mathrm{FOR}_{\mathrm{mbC}} \cup \{\chi A : A \in \mathrm{FOR}^{\sim\circ}_{\mathrm{mbC}}\} \cup \{\neg\chi A : A \in \mathrm{FOR}^{\sim\circ}_{\mathrm{mbC}}\}$$

(1) $A \to (B \to A)$

(2) $(A \to B) \to (((A \to (B \to C)) \to (A \to C))$

(3) $A \to (B \to (A \wedge B))$

(4) $(A \wedge B) \to A$

(5) $(A \wedge B) \to B$

(6) $A \to (A \vee B)$

(7) $B \to (A \vee B)$

(8) $(A \to C) \to ((B \to C) \to (A \vee B \to C))$

(9) $A \vee (A \to B)$

(10) $A \vee \neg A$

(11) $A \to (\neg A \to B)$

(12) $A \vee {\sim} A$

(13) $\circ A \to (A \to ({\sim} A \to B))$

(**MP**) If $\vdash_{\mathsf{mbC}} A$ and $\vdash_{\mathsf{mbC}} A \to B$, then $\vdash_{\mathsf{mbC}} B$

mbC-semivaluation

An mbC-semivaluation is a function $v : \mathrm{FOR}^{\mathsf{mbC}} \longrightarrow \{0, 1\}$ which behaves in a standard way in the case of classical connectives and the following conditions are satisfied:

($\sim$) if $v(\sim A) = 0$, then $v(A) = 1$;

($\circ$) if $v(\circ A) = 1$, then either $v(A) = 0$ or $v(\sim A) = 0$.

Questions of $\mathcal{L}^{?}_{mbC}$

An atomic declarative formula (sequent) of $\mathcal{L}^{?}_{\mathsf{mbC}}$

$$\Gamma \vdash \Delta$$

where Γ and Δ are finite, non-empty, sequences of formulas of $\mathcal{L}^{+}_{\mathsf{mbC}}$.

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where Γ and Δ are finite, non-empty, sequences of formulas of $\mathcal{L}^{+}_{\mathsf{mbC}}$.

Questions of $\mathcal{L}^{?}_{\mathsf{mbC}}$

$$?(\Phi)$$

where Φ is a finite, non-empty sequence of sequents of $\mathcal{L}^{?}_{\mathsf{mbC}}$.

α	α_1	α_2	β	β_1	β_2
$A \wedge B$	A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B
$\neg(A \rightarrow B)$	A	$\neg B$	$A \rightarrow B$	$\neg A$	B

$$\frac{?(\Phi;\Gamma,\alpha,\Gamma'\vdash\Delta;\Psi)}{?(\Phi;\Gamma,\alpha_1,\alpha_2,\Gamma'\vdash\Delta;\Psi)}\ \mathbf{L}_\alpha \qquad \frac{?(\Phi;\Gamma\vdash\Delta,\alpha,\Delta';\Psi)}{?(\Phi;\Gamma\vdash\Delta,\alpha_1,\Delta';\Gamma\vdash\Delta,\alpha_2,\Delta';\Psi)}\ \mathbf{R}_\alpha$$

$$\frac{?(\Phi;\Gamma,\beta,\Gamma'\vdash\Delta;\Psi)}{?(\Phi;\Gamma,\beta_1,\Gamma'\vdash\Delta;\Gamma,\beta_2,\Gamma'\vdash\Delta;\Psi)}\ \mathbf{L}_\beta \qquad \frac{?(\Phi;\Gamma\vdash\Delta,\beta,\Delta';\Psi)}{?(\Phi;\Gamma\vdash\Delta,\beta_1,\beta_2,\Delta';\Psi)}\ \mathbf{R}_\beta$$

$$\frac{?(\Phi;\Gamma,\neg\neg A,\Gamma'\vdash\Delta;\Psi)}{?(\Phi;\Gamma,A,\Gamma'\vdash\Delta;\Psi)}\ \mathbf{L}_{\neg\neg} \qquad \frac{?(\Phi;\Gamma\vdash\Delta,\neg\neg A,\Delta';\Psi)}{?(\Phi;\Gamma\vdash\Delta,A,\Delta';\Psi)}\ \mathbf{R}_{\neg\neg}$$

$$\frac{?(\Phi; \Gamma, \sim A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, \neg A, \Gamma' \vdash \Delta; \Gamma, \chi \sim A, \Gamma' \vdash \Delta : \Psi)}\ \mathbf{L}_{\sim}$$

$$\frac{?(\Phi; \Gamma \vdash \Delta, \sim A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, \neg A, \chi \sim A, \Delta'; \Psi)}\ \mathbf{R}_{\sim}$$

$$\frac{?(\Phi; \Gamma, \neg \sim A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, A, \neg \chi \sim A, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_{\neg\sim}$$

$$\frac{?(\Phi; \Gamma \vdash \Delta, \neg \sim A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, A, \Delta'; \Gamma \vdash \Delta, \neg \chi \sim A, \Delta'; \Psi)}\ \mathbf{R}_{\neg\sim}$$

$$\frac{?(\Phi; \Gamma, \circ A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, \neg A, \chi\circ A, \Gamma' \vdash \Delta; \Gamma, \neg{\sim}A, \chi\circ A, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_{\circ}$$

$$\frac{?(\Phi; \Gamma \vdash \Delta, \circ A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, \chi\circ A, \Delta'; \Gamma \vdash \Delta, \neg A, \neg{\sim}A, \Delta'; \Psi)}\ \mathbf{R}_{\circ}$$

$$\frac{?(\Phi; \Gamma, \neg\circ A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, A, {\sim}A, \Gamma' \vdash \Delta; \Gamma, \neg\chi\circ A, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_{\neg\circ}$$

$$\frac{?(\Phi; \Gamma \vdash \Delta, \neg\circ A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, A, \neg\chi\circ A, \Delta'; \Gamma \vdash \Delta, {\sim}A, \neg\chi\circ A, \Delta'; \Psi)}\ \mathbf{R}_{\neg\circ}$$

Using IEL check whether the following formulas are mbC tautologies:

1. $(\circ p \wedge (p \wedge \sim p)) \rightarrow q$
2. $\sim (p \wedge q) \rightarrow \sim (q \wedge p)$
3. $\circ p \rightarrow \sim (p \wedge \sim p)$
4. $p \wedge \sim p \rightarrow \sim \circ p$

Example

Knowledge base:

$$\Gamma = \langle p \to (q \to r), \neg(q \to \sim s)\rangle$$

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What we want to derive:

$$\Delta = \langle z \rangle$$

Knowledge base:

$$\Gamma = \langle p \to (q \to r), \neg(q \to \sim s)\rangle$$

What we want to derive:

$$\Delta = \langle z \rangle$$

The question arises:

$$?(\Gamma \vdash \Delta)$$

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

Example

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\frac{?(p \to (q \to r), \neg(q \to \sim s) \vdash z)}{?(q, \neg\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_\alpha$$

$$\frac{?(\Phi; \Gamma, \alpha, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \alpha_1, \alpha_2, \Gamma, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_\alpha$$

Example

$$\cfrac{\cfrac{?(p \to (q \to r), \neg(q \to \sim s) \vdash z)}{?(q, \neg\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_{\alpha}}{?(q, s, \neg\chi\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_{\neg\sim}$$

$$\frac{?(\Phi; \Gamma, \neg\sim A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, A, \neg\chi\sim A, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_{\neg\sim}$$

Example

$$\dfrac{\dfrac{\dfrac{?(p \to (q \to r), \neg(q \to \sim s) \vdash z)}{?(q, \neg\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_\alpha}{?(q, s, \neg\chi\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_{\neg\sim}}{?(\neg p, q, s, \neg\chi\sim s \vdash z\ ;\ q \to r, q, s, \neg\chi\sim s \vdash z)}\ \mathbf{L}_\beta$$

$$\dfrac{?(\Phi; \Gamma, \beta, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \beta_1, \Gamma, \Gamma' \vdash \Delta; \beta_2, \Gamma, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_\beta$$

$$\dfrac{\dfrac{\dfrac{\dfrac{?(p \to (q \to r), \neg(q \to \sim s) \vdash z)}{?(q, \neg\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_{\alpha}}{?(q, s, \neg\chi\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_{\neg\sim}}{?(\neg p, q, s, \neg\chi\sim s \vdash z\ ;\ q \to r, q, s, \neg\chi\sim s \vdash z)}\ \mathbf{L}_{\beta}}{?(\neg p, q, s, \neg\chi\sim s \vdash z\ ;\ \neg q, q, s, \neg\chi\sim s \vdash z\ ;\ r, q, s, \neg\chi\sim s \vdash z)}\ \mathbf{L}_{\beta}$$

$$\dfrac{?(\Phi; \Gamma, \beta, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \beta_1, \Gamma, \Gamma' \vdash \Delta; \beta_2, \Gamma, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_{\beta}$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{?(p \to (q \to r), \neg(q \to \sim s) \vdash z)}{?(q, \neg\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_\alpha}{?(q, s, \neg\chi\sim s, p \to (q \to r) \vdash z)}\ \mathbf{L}_{\neg\sim}}{?(\neg p, q, s, \neg\chi\sim s \vdash z\ ;\ q \to r, q, s, \neg\chi\sim s \vdash z)}\ \mathbf{L}_\beta}{?(\neg p, q, s, \neg\chi\sim s \vdash z\ ;\ \boxed{\neg q, q, s, \neg\chi\sim s \vdash z}\ ;\ r, q, s, \neg\chi\sim s \vdash z)}\ \mathbf{L}_\beta$$

$$\dfrac{\dfrac{\dfrac{\dfrac{?(p \to (q \to r), \neg(q \to \sim s) \vdash z)}{?(q, \neg\sim s, p \to (q \to r) \vdash z)}\;\mathbf{L}_{\alpha}}{?(q, s, \neg\chi\sim s, p \to (q \to r) \vdash z)}\;\mathbf{L}_{\neg\sim}}{?(\neg p, q, s, \neg\chi\sim s \vdash z \;;\; q \to r, q, s, \neg\chi\sim s \vdash z)}\;\mathbf{L}_{\beta}}{?(\neg p, q, s, \neg\chi\sim s \vdash z \;;\; \neg q, q, s, \neg\chi\sim s \vdash z \;;\; r, q, s, \neg\chi\sim s \vdash z)}\;\mathbf{L}_{\beta}$$

Example

$$?(p \to (q \to r), \neg(q \to {\sim} s) \vdash z)$$

$$\vdots$$

$$?(\neg p, q, s, \neg\chi{\sim}s \vdash z \;;\; \neg q, q, s, \neg\chi{\sim}s \vdash z \;;\; r, q, s, \neg\chi{\sim}s \vdash z)$$

Example

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\ \neg p, q, s, \neg\chi\!\sim\! s \vdash z\ ;\ \neg q, q, s, \neg\chi\!\sim\! s \vdash z\ ;\ r, q, s, \neg\chi\!\sim\! s \vdash z\ )$$

A_1 for $\neg p, q, s, \neg\chi\!\sim\! s \vdash z$:

- p, $\neg q$, $\neg s$
- $\neg p \to z$, $q \to z$, $s \to z$

Example

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\neg p, q, s, \neg\chi \sim s \vdash z \;;\; \neg q, q, s, \neg\chi \sim s \vdash z \;;\; r, q, s, \neg\chi \sim s \vdash z)$$

A_1 for $\neg p, q, s, \neg\chi \sim s \vdash z$:

- p, $\neg q$, $\neg s$
- $\neg p \to z$, $q \to z$, $s \to z$

A_2 for $r, q, s, \neg\chi \sim s \vdash z$:

- $\neg r$, $\neg q$, $\neg s$
- $r \to z$, $q \to z$, $s \to z$

Example

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\boxed{\neg p, q, s, \neg\chi \sim s \vdash z}\ ;\ \neg q, q, s, \neg\chi \sim s \vdash z\ ;\ \boxed{r, q, s, \neg\chi \sim s \vdash z})$$

A_1 for $\neg p, q, s, \neg\chi \sim s \vdash z$:

- p, $\neg q$, $\neg s$
- $\neg p \to z$, $q \to z$, $s \to z$

A_2 for $r, q, s, \neg\chi \sim s \vdash z$:

- $\neg r$, $\neg q$, $\neg s$
- $r \to z$, $q \to z$, $s \to z$

$$H = A_1 \wedge A_2$$

Rules for answering questions of $\mathbb{E}^{\mathsf{CPL}}$

$$\frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta''\ ;\ \Psi)}{\bar{l}}\ \mathbf{R}^1_{abd}$$

$$\frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta'', k, \Theta'''\ ;\ \Psi)}{l \to k}\ \mathbf{R}^2_{abd}$$

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', \sim p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{p}\ \mathbf{R}^{3}_{abd}$$

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{\sim p}\ \mathbf{R}^{4}_{abd}$$

$$\frac{?(\Phi\ ;\ \Theta, p, \Theta', \sim p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{\circ p}\ \mathbf{R}^{5}_{abd}$$

$$\frac{?(\Phi\ ;\ \Theta, \sim p, \Theta', \circ p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{p}\ \mathbf{R}^{3^*}_{abd}$$

$$\frac{?(\Phi\ ;\ \Theta, p, \Theta', \circ p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{\sim p}\ \mathbf{R}^{4^*}_{abd}$$

$$\frac{?(\Phi\ ;\ \Theta, \sim p, \Theta', p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{\circ p}\ \mathbf{R}^{5^*}_{abd}$$

Properties of abductive hypotheses

1. Consistency: $\Gamma \cup \{H\}$ is consistent.
2. Significance: $H \nvdash_{\mathsf{CPL}} \Delta$.

Let Γ be a sequence of formulas of $\mathcal{L}^{+}_{\mathsf{mbC}}$. By a *downward saturated set* (or *Hintikka set*) corresponding to a sequence Γ we mean a set $\mathfrak{U}_\Gamma$, which fulfils the following conditions:

1. if A is a term of Γ, then $A \in \mathfrak{U}_\Gamma$,
2. if $\alpha \in \mathfrak{U}_\Gamma$, then $\alpha_1 \in \mathfrak{U}_\Gamma$ and $\alpha_2 \in \mathfrak{U}_\Gamma$,
3. if $\beta \in \mathfrak{U}_\Gamma$, then $\beta_1 \in \mathfrak{U}_\Gamma$ or $\beta_2 \in \mathfrak{U}_\Gamma$,
4. if $\neg\neg A \in \mathfrak{U}_\Gamma$, then $A \in \mathfrak{U}_\Gamma$,
5. if $\sim A \in \mathfrak{U}_\Gamma$, then $\neg A \in \mathfrak{U}_\Gamma$ or $\chi{\sim}A \in \mathfrak{U}_\Gamma$,
6. if $\neg \sim A \in \mathfrak{U}_\Gamma$, then $A \in \mathfrak{U}_\Gamma$ and $\neg\chi{\sim}A \in \mathfrak{U}_\Gamma$,
7. if $\circ A \in \mathfrak{U}_\Gamma$, then $(\neg A \in \mathfrak{U}_\Gamma$ and $\chi\circ A \in \mathfrak{U}_\Gamma)$ or $(\neg{\sim}A \in \mathfrak{U}_\Gamma$ and $\chi\circ A \in \mathfrak{U}_\Gamma)$,
8. if $\neg \circ A \in \mathfrak{U}_\Gamma$, then $(A \in \mathfrak{U}_\Gamma$ and $\sim A \in \mathfrak{U}_\Gamma)$ or $\neg\chi\circ A \in \mathfrak{U}_\Gamma$,
9. nothing more belongs to $\mathfrak{U}_\Gamma$ except those formulas which enter $\mathfrak{U}_\Gamma$ on the grounds of conditions 1–8.

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3. if $\beta \in \mathfrak{U}_\Gamma$, then $\beta_1 \in \mathfrak{U}_\Gamma$ or $\beta_2 \in \mathfrak{U}_\Gamma$,
4. if $\neg\neg A \in \mathfrak{U}_\Gamma$, then $A \in \mathfrak{U}_\Gamma$,
5. if $\sim A \in \mathfrak{U}_\Gamma$, then $\neg A \in \mathfrak{U}_\Gamma$ or $\chi{\sim}A \in \mathfrak{U}_\Gamma$,
6. if $\neg \sim A \in \mathfrak{U}_\Gamma$, then $A \in \mathfrak{U}_\Gamma$ and $\neg\chi{\sim}A \in \mathfrak{U}_\Gamma$,
7. if $\circ A \in \mathfrak{U}_\Gamma$, then $(\neg A \in \mathfrak{U}_\Gamma$ and $\chi\circ A \in \mathfrak{U}_\Gamma)$ or $(\neg{\sim}A \in \mathfrak{U}_\Gamma$ and $\chi\circ A \in \mathfrak{U}_\Gamma)$,
8. if $\neg \circ A \in \mathfrak{U}_\Gamma$, then $(A \in \mathfrak{U}_\Gamma$ and $\sim A \in \mathfrak{U}_\Gamma)$ or $\neg\chi\circ A \in \mathfrak{U}_\Gamma$,
9. nothing more belongs to $\mathfrak{U}_\Gamma$ except those formulas which enter $\mathfrak{U}_\Gamma$ on the grounds of conditions 1–8.

Consistency property:

$$\mathfrak{U}^c_\Gamma = \{\mathfrak{U}^1_\Gamma, \ldots, \mathfrak{U}^n_\Gamma\}$$

Consistency restrictions

$$\frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta''\ ;\ \Psi)}{\bar{l}}\ \mathbf{R}^1_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $l \notin \mathfrak{U}_\Gamma$.

$$\frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta''\ ;\ \Psi)}{\bar{l}}\ \mathbf{R}^1_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $l \notin \mathfrak{U}_\Gamma$.

$$\frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta'', k, \Theta'''\ ;\ \Psi)}{l \rightarrow k}\ \mathbf{R}^2_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $l \notin \mathfrak{U}_\Gamma$ or $\overline{k} \notin \mathfrak{U}_\Gamma$.

Consistency restrictions

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', {\sim} p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{p}\ \mathbf{R}^{3}_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $\circ p \notin \mathfrak{U}_\Gamma$ or ${\sim} p \notin \mathfrak{U}_\Gamma$.

Consistency restrictions

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', \sim p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{p}\ \mathbf{R}^3_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $\circ p \notin \mathfrak{U}_\Gamma$ or $\sim p \notin \mathfrak{U}_\Gamma$.

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{\sim p}\ \mathbf{R}^4_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $\circ p \notin \mathfrak{U}_\Gamma$ or $p \notin \mathfrak{U}_\Gamma$.

Consistency restrictions

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', {\sim} p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{p}\ \mathbf{R}^3_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $\circ p \notin \mathfrak{U}_\Gamma$ or ${\sim} p \notin \mathfrak{U}_\Gamma$.

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{{\sim} p}\ \mathbf{R}^4_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $\circ p \notin \mathfrak{U}_\Gamma$ or $p \notin \mathfrak{U}_\Gamma$.

$$\frac{?(\Phi\ ;\ \Theta, p, \Theta', {\sim} p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{\circ p}\ \mathbf{R}^5_{abd}$$

There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $p \notin \mathfrak{U}_\Gamma$ or ${\sim} p \notin \mathfrak{U}_\Gamma$.

Dual downward saturated set

Let Δ be a sequence of formulas of $\mathcal{L}^{+}_{\mathsf{mbC}}$. By a *dual downward saturated set* (or *dual Hintikka set*) corresponding to a sequence Δ we mean a set $\mathfrak{W}_{\Delta}$, which fulfils the following conditions:

1. if A is a term of Δ, then $A \in \mathfrak{W}_{\Delta}$,
2. if $\alpha \in \mathfrak{W}_{\Delta}$, then $\alpha_1 \in \mathfrak{W}_{\Delta}$ or $\alpha_2 \in \mathfrak{W}_{\Delta}$,
3. if $\beta \in \mathfrak{W}_{\Delta}$, then $\beta_1 \in \mathfrak{W}_{\Delta}$ and $\beta_2 \in \mathfrak{W}_{\Delta}$,
4. if $\neg\neg A \in \mathfrak{W}_{\Delta}$, then $A \in \mathfrak{W}_{\Delta}$,
5. if $\sim A \in \mathfrak{W}_{\Delta}$, then $\neg A \in \mathfrak{W}_{\Delta}$ and $\chi\!\sim A \in \mathfrak{W}_{\Delta}$,
6. if $\neg \sim A \in \mathfrak{W}_{\Delta}$, then $A \in \mathfrak{W}_{\Delta}$ or $\neg\chi\!\sim A \in \mathfrak{W}_{\Delta}$,
7. if $\circ A \in \mathfrak{W}_{\Delta}$, then $\chi\circ A \in \mathfrak{W}_{\Delta}$ or ($\neg A \in \mathfrak{W}_{\Delta}$ and $\neg\!\sim\! A \in \mathfrak{W}_{\Delta}$),
8. if $\neg\circ A \in \mathfrak{W}_{\Delta}$, then ($A \in \mathfrak{W}_{\Delta}$ and $\neg\chi\circ A \in \mathfrak{W}_{\Delta}$) or ($\sim A \in \mathfrak{W}_{\Delta}$ and $\neg\chi\circ A \in \mathfrak{W}_{\Delta}$),
9. nothing more belongs to $\mathfrak{W}_{\Delta}$ except those formulas which enter $\mathfrak{W}_{\Delta}$ on the grounds of conditions 1–6.

Dual downward saturated set

Let Δ be a sequence of formulas of $\mathcal{L}^{+}_{\mathsf{mbC}}$. By a *dual downward saturated set* (or *dual Hintikka set*) corresponding to a sequence Δ we mean a set $\mathfrak{W}_\Delta$, which fulfils the following conditions:

1. if A is a term of Δ, then $A \in \mathfrak{W}_\Delta$,
2. if $\alpha \in \mathfrak{W}_\Delta$, then $\alpha_1 \in \mathfrak{W}_\Delta$ or $\alpha_2 \in \mathfrak{W}_\Delta$,
3. if $\beta \in \mathfrak{W}_\Delta$, then $\beta_1 \in \mathfrak{W}_\Delta$ and $\beta_2 \in \mathfrak{W}_\Delta$,
4. if $\neg\neg A \in \mathfrak{W}_\Delta$, then $A \in \mathfrak{W}_\Delta$,
5. if $\sim A \in \mathfrak{W}_\Delta$, then $\neg A \in \mathfrak{W}_\Delta$ and $\chi\sim A \in \mathfrak{W}_\Delta$,
6. if $\neg \sim A \in \mathfrak{W}_\Delta$, then $A \in \mathfrak{W}_\Delta$ or $\neg\chi\sim A \in \mathfrak{W}_\Delta$,
7. if $\circ A \in \mathfrak{W}_\Delta$, then $\chi\circ A \in \mathfrak{W}_\Delta$ or ($\neg A \in \mathfrak{W}_\Delta$ and $\neg\sim A \in \mathfrak{W}_\Delta$),
8. if $\neg\circ A \in \mathfrak{W}_\Delta$, then ($A \in \mathfrak{W}_\Delta$ and $\neg\chi\circ A \in \mathfrak{W}_\Delta$) or ($\sim A \in \mathfrak{W}_\Delta$ and $\neg\chi\circ A \in \mathfrak{W}_\Delta$),
9. nothing more belongs to $\mathfrak{W}_\Delta$ except those formulas which enter $\mathfrak{W}_\Delta$ on the grounds of conditions 1–6.

Non-validity property:

$$\mathfrak{W}^{nv}_\Delta = \{\mathfrak{W}^1_\Delta, \ldots, \mathfrak{W}^n_\Delta\}$$

Significance restrictions

$$\frac{?(\Phi \ ; \ \Theta, l, \Theta' \vdash \Theta'' \ ; \ \Psi)}{\bar{l}} \ \mathbf{R}^1_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $\bar{l} \notin \mathfrak{W}_\Delta$.

Significance restrictions

$$\frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta''\ ;\ \Psi)}{\bar{l}}\ \mathbf{R}^1_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $\bar{l} \notin \mathfrak{W}_\Delta$.

$$\frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta'', k, \Theta'''\ ;\ \Psi)}{l \to k}\ \mathbf{R}^2_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $\bar{l} \notin \mathfrak{W}_\Delta$ or $k \notin \mathfrak{W}_\Delta$.

Significance restrictions

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', \sim p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{p}\ \mathbf{R}^3_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $p \notin \mathfrak{W}_\Delta$.

Significance restrictions

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', \sim p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{p}\ \mathbf{R}^3_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $p \notin \mathfrak{W}_\Delta$.

$$\frac{?(\Phi\ ;\ \Theta, \circ p, \Theta', p, \Theta'' \vdash \Theta'''\ ;\ \Psi)}{\sim p}\ \mathbf{R}^4_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $\sim p \notin \mathfrak{W}_\Delta$.

Significance restrictions

$$\frac{?(\Phi \ ; \ \Theta, \circ p, \Theta', \sim p, \Theta'' \vdash \Theta''' \ ; \ \Psi)}{p} \ \mathbf{R}^3_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $p \notin \mathfrak{W}_\Delta$.

$$\frac{?(\Phi \ ; \ \Theta, \circ p, \Theta', p, \Theta'' \vdash \Theta''' \ ; \ \Psi)}{\sim p} \ \mathbf{R}^4_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $\sim p \notin \mathfrak{W}_\Delta$.

$$\frac{?(\Phi \ ; \ \Theta, p, \Theta', \sim p, \Theta'' \vdash \Theta''' \ ; \ \Psi)}{\circ p} \ \mathbf{R}^5_{abd}$$

There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $\circ p \notin \mathfrak{W}_\Delta$.

Example — continuation

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\neg p, q, s, \neg\chi{\sim}s \vdash z \;;\; \neg q, q, s, \neg\chi{\sim}s \vdash z \;;\; r, q, s, \neg\chi{\sim}s \vdash z)$$

Example — continuation

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\boxed{\neg p, q, s, \neg\chi\sim s \vdash z} \; ; \; \neg q, q, s, \neg\chi\sim s \vdash z \; ; \; \boxed{r, q, s, \neg\chi\sim s \vdash z})$$

$\mathfrak{U}^c_\Gamma$:

- $\mathfrak{U}^1_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi\sim s, \neg p, q \to r, r\}$
- $\mathfrak{U}^2_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi\sim s, \neg p\}$
- $\mathfrak{U}^3_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi\sim s, q \to r, r\}$

Example — continuation

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\neg p, q, s, \neg\chi{\sim}s \vdash z \;;\; \neg q, q, s, \neg\chi{\sim}s \vdash z \;;\; r, q, s, \neg\chi{\sim}s \vdash z)$$

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- $\mathfrak{U}^3_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, q \to r, r\}$

$\mathfrak{W}^{nv}_\Delta$:

- $\mathfrak{W}^1_\Delta = \{z\}$

Abductive hypothesis

1. Choose A_i.
2. Leave in $\mathfrak{U}_\Gamma^c$ only those $\mathfrak{U}_\Gamma$ that are consistent with A_i.
3. If there are still open sequents, then choose A_j...

Example — continuation

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\boxed{\neg p, q, s, \neg\chi{\sim}s \vdash z}\ ;\ \neg q, q, s, \neg\chi{\sim}s \vdash z\ ;\ \boxed{r, q, s, \neg\chi{\sim}s \vdash z})$$

$\mathfrak{U}^c_\Gamma$:

- $\mathfrak{U}^1_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, \neg p, q \to r, r\}$
- $\mathfrak{U}^2_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, \neg p\}$
- $\mathfrak{U}^3_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, q \to r, r\}$

Example — continuation

$$?(p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s) \vdash z)$$

$$\vdots$$

$$?(\neg p, q, s, \neg\chi\sim s \vdash z \;;\; \neg q, q, s, \neg\chi\sim s \vdash z \;;\; r, q, s, \neg\chi\sim s \vdash z)$$

$A_1 = p$

$\mathfrak{U}^c_\Gamma$:

- $\mathfrak{U}^1_\Gamma = \{p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s), q, s, \neg\chi\sim s, \neg p, q \rightarrow r, r\}$
- $\mathfrak{U}^2_\Gamma = \{p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s), q, s, \neg\chi\sim s, \neg p\}$
- $\mathfrak{U}^3_\Gamma = \{p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s), q, s, \neg\chi\sim s, q \rightarrow r, r\}$

Example — continuation

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\boxed{\neg p, q, s, \neg\chi{\sim}s \vdash z} \; ; \; \neg q, q, s, \neg\chi{\sim}s \vdash z \; ; \; \boxed{r, q, s, \neg\chi{\sim}s \vdash z})$$

$A_1 = p$

$\mathfrak{U}^c_\Gamma$:

- $\mathfrak{U}^1_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, \neg p, q \to r, r\}$
- $\mathfrak{U}^2_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, \neg p\}$
- $\mathfrak{U}^3_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, q \to r, r\}$

Example — continuation

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\neg p, q, s, \neg\chi{\sim}s \vdash z\ ;\ \neg q, q, s, \neg\chi{\sim}s \vdash z\ ;\ r, q, s, \neg\chi{\sim}s \vdash z)$$

$A_1 = p$
$A_2 = q \to z$

$\mathfrak{U}^c_\Gamma$:

- $\mathfrak{U}^1_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, \neg p, q \to r, r\}$
- $\mathfrak{U}^2_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, \neg p\}$
- $\mathfrak{U}^3_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, q \to r, r\}$

Example — continuation

$$?(p \to (q \to r), \neg(q \to \sim s) \vdash z)$$

$$\vdots$$

$$?(\neg p, q, s, \neg\chi{\sim}s \vdash z \; ; \; \neg q, q, s, \neg\chi{\sim}s \vdash z \; ; \; r, q, s, \neg\chi{\sim}s \vdash z)$$

$A_1 = p$
$A_2 = q \to z$

$$H = p \land (q \to z)$$

$\mathfrak{U}^c_\Gamma$:

- $\mathfrak{U}^1_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, \neg p, q \to r, r\}$
- $\mathfrak{U}^2_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, \neg p\}$
- $\mathfrak{U}^3_\Gamma = \{p \to (q \to r), \neg(q \to \sim s), q, s, \neg\chi{\sim}s, q \to r, r\}$

Find (if possible) consistent and significant abductive hypothes es for the following problems:

1. $\Gamma = \langle p \rightarrow q \rangle$, $\Delta = \langle \sim q \rightarrow \sim p \rangle$
2. $\Gamma = \langle p \rightarrow q, q \rightarrow r \rangle$, $\Delta = \langle \sim r \rightarrow \sim p \rangle$
3. $\Gamma = \langle p \rightarrow q, p \rightarrow \sim q \rangle$, $\Delta = \langle \sim p \rangle$
4. $\Gamma = \langle \sim p \rightarrow q, \sim p \rightarrow \sim q \rangle$, $\Delta = \langle p \rangle$
5. $\Gamma = \langle \circ p, p, q \rightarrow p \rangle$, $\Delta = \langle r \rangle$
6. $\Gamma = \langle p \rightarrow (q \rightarrow r), \sim (q \rightarrow \sim s), \circ (q \rightarrow \sim s), \circ s \rangle$, $\Delta = \langle z \rangle$