Abductive Question-Answer System (AQAS) for Classical Propositional Logic

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Abstract. We propose a new approach to modelling abductive reasoning by means of an abductive question-answer system. We introduce the concept of an abductive question which is the starting point of abductive reasoning. The result of applying the question processing procedure is a question, which is simpler than the initial one. AQAS generates abductive hypotheses that fulfil certain criteria in one step, i.e. processes of generation and evaluation of abductive hypotheses are integrated.

Keywords: Logic of questions, Inferential Erotetic Logic, erotetic calculi, abduction

Introduction

The general schema of abductive reasoning could be described as follows: given the known rule *if H, then A* and an observation of A, infer H [15]. In other words, we can say that products of abductive procedures serve as a filler of the cognitive gap when some puzzling phenomenon is observed [7]. These properties account for the fact that abductive reasoning is used to solve problems in science (e.g. explanation of new observations), real life (e.g. diagnosis in medicine), and also in fiction (e.g. detective Sherlock Holmes) [12,15,16,18,19].

There are abductive procedures designed for Classical Propositional Calculus (e.g. [1]), other propositional logics (e.g. [13]) and first-order logic (e.g. [14], [9]). Different kinds of approaches to the problem use different proof methods for abductive procedures (for example [1] used analytic tableaux, [13] sequent calculi and [14] proof method of adaptive logics).

The aim of this article is to propose a model of abductive reasoning based on logic of questions. We interpret the abductive problem as an abductive question: *what should be added to the knowledge base Γ in order to be able to derive a fact φ?*, where φ is not derivable from Γ. Our proposal is based on a decomposition of the initial question an agent asks himself when he encounters an abductive problem. Therefore Wiśniewski's method of Socratic Proofs (see for example [21]) is being used as a main proof theoretical mechanism. It is a tool developed on the grounds of Wiśniewski's Inferential Erotetic Logic (IEL) (see [23,22]).

In general, approaches to the problem of formal specification of the abductive reasoning may differ in many respects and, in our opinion, one of the most interesting features of these procedures pertains to the relation between the generation and evaluation of abductive hypotheses. On the one hand, there are procedures which generate a large set of abductive hypotheses and then select 'good' hypotheses from this set, i.e. hypotheses which fulfil certain criteria [1,10]. On the other hand, one may think of abductive reasoning in such a way that the creation and evaluation of hypotheses are strongly intertwined: only those hypotheses are generated which are permitted given a certain set of criteria. The latter seems to us more natural. In the real-life as well as in scientific reasoning people do not waste time on the creation of hypotheses that may or may not be 'good'. They are interested only in 'good' hypotheses [12].

We consider algorithmic account of abduction with the following ingredients: Classical Propositional Logic as a basic logic and the method of Socratic Proofs [2,11,21] as a proof method.[1] As we mentioned, we exploit the approach where generation and evaluation of the hypotheses is conducted in one step. Our goal here is to introduce a new model of abductive reasoning based on IEL, therefore any detailed comparison concerning efficiency of the abductive procedures with the existing approaches will not take place in this article[2].

1 Question processing

Since in our model abductive reasoning is triggered by an abductive question, we need some techniques enabling question processing. For that purpose we use some concepts and tools of IEL.

1.1 Language of IEL

We use the language $\mathcal{L}_{\mathsf{CPL}}$ of Classical Propositional Logic defined as usual. The language $\mathcal{L}^{?}_{\vdash\mathsf{CPL}}$ is an object-level language in which our erotetic calculi will

[1] Urbański and Wiśniewski [20] proposed a mechanism which enables to obtain abductive hypotheses in the form of law-like statements. The basis of the mechanism is similar as we use here. However, the two approaches differ when results of the abductive procedures are concerned. What is more, Urbański and Wiśniewski put it explicitly at the beginning of their article that they will not consider problem of the evaluation of abductive hypotheses.

[2] However some remarks should be made at this point. In the well-known *Abductive Logic Programming* (ALP) framework (on the propositional level) it is assumed that the set of abductive hypotheses (the set of abducibles) is known before abductive reasoning is triggered. Then, using integrity constraints and information from the knowledge base it can be figured out which hypotheses are good. Moreover, abductive hypotheses can be only of the form of atomic formulas. In AQAS the set of abductive hypotheses is not known before the initial question is transformed and abductive hypotheses can be literals as well as formulas of the form of implication. We think that the novelty of our approach lays in the fact that the concept of abductive hypothesis is defined in a more general way.

be worded. The meaningful expressions of the language $\mathcal{L}^?_{\vdash\mathsf{CPL}}$ belong to two disjoint sets. The first one consists of *declarative well-formed formulas* (d-wffs for short). The second one is the set of *erotetic well-formed formulas* (e-wffs or simply questions).

To obtain the vocabulary of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$ we add to the vocabulary of $\mathcal{L}_{\mathsf{CPL}}$ the following signs: $\vdash$ (turnstile, intuitively stands for derivability relation in CPL), ? (a question mark for constructing questions of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$) and , (comma), ; (semicolon).

Definition 1. *Let Γ, Δ be finite, non-empty, sequences of formulas of $\mathcal{L}_{\mathsf{CPL}}$. An* atomic declarative formula *of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$ or* sequent *is of the following form:*

$$\Gamma \vdash \Delta$$

Definition 2. *Questions of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$ have the following form:*

$$?(\Phi)$$

where Φ is a finite, non-empty sequence of sequents of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$.

We use commas for separating formulas in sequents and semicolons for separating sequents in sequences. The following expressions are thus questions of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$: $?(\neg(p \to q), \neg r \vdash r \wedge q)$, $?(p, \neg r \vdash p \vee r \ ; \ q \wedge \neg r \vdash r \ ; \ p \vdash p)$.

The intuitive meaning of a sequent $\Gamma \vdash \Delta$ is given in terms of multiple-conclusion entailment (mc-entailment for short): if all formulas in Γ are true, then at least one formula in Δ is true also (in symbols: $\Gamma \Vdash_{\mathsf{CPL}}$).

However '$\vdash$' is an object level expression of the language $\mathcal{L}^?_{\vdash\mathsf{CPL}}$ that should not be confused with the metalanguage expression '$\vdash_{\mathsf{CPL}}$' which is a syntactical or semantical consequence relation generated by CPL or with the mc-entailment relation '$\Vdash_{\mathsf{CPL}}$'.

If $\Gamma \Vdash_{\mathsf{CPL}} \Delta$ then we say that the sequent $\Gamma \vdash \Delta$ of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$ is *closed*, otherwise it is *open*. If a sequent consists of literals only, it is called an *atomic sequent*. If $\Psi = \langle \phi_1, \ldots, \phi_n \rangle$ and for each i $(1 \leq i \leq n)$, ϕ_i is an atomic sequent, then the question $?(\Psi)$ is called a *minimal question*.

Definition 3 (Abductive question). *An* abductive question *(or* abductive problem*) has the following form:*

$$?(\Psi)$$

where Ψ is a non-empty sequence of sequents such that at least one term of Ψ is an open sequent of $\mathcal{L}^?_{\vdash\mathsf{CPL}}$. If $\Psi = \langle \phi \rangle$ is a one-term sequence, then the question $?(\Psi)$ is called simple*. If ϕ is also an open sequent, then $?(\Psi)$ is an* simple abductive question*. If $\Psi = \langle \phi_1, \ldots, \phi_n \rangle$ and for each i $(1 \leq i \leq n)$, ϕ_i is an atomic sequent, then the question $?(\Psi)$ is called a* minimal abductive question*.*

Intuitively, given an open sequent $\Gamma \vdash \Delta$, the antecedent Γ represents a knowledge base (such that some formulas/pieces of information can be repeated), which is used by an agent to 'explain' (derive) the data represented by Δ.

Table 1. Rules of $\mathbb{E}^{\mathsf{CPL}}$

$$\frac{?(\Phi; \Gamma, \alpha, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, \alpha_1, \alpha_2, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_\alpha \qquad \frac{?(\Phi; \Gamma \vdash \Delta, \alpha, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, \alpha_1, \Delta'; \Gamma \vdash \Delta, \alpha_2, \Delta'; \Psi)}\ \mathbf{R}_\alpha$$

$$\frac{?(\Phi; \Gamma, \beta, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, \beta_1, \Gamma' \vdash \Delta; \Gamma, \beta_2, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_\beta \qquad \frac{?(\Phi; \Gamma \vdash \Delta, \beta, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, \beta_1, \beta_2, \Delta'; \Psi)}\ \mathbf{R}_\beta$$

$$\frac{?(\Phi; \Gamma, \neg\neg A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, A, \Gamma' \vdash \Delta; \Psi)}\ \mathbf{L}_{\neg\neg} \qquad \frac{?(\Phi; \Gamma \vdash \Delta, \neg\neg A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, A, \Delta'; \Psi)}\ \mathbf{R}_{\neg\neg}$$

1.2 Erotetic rules of inference

Let $?(\Gamma \vdash \Delta)$ be an abductive question. The formulas which belong to Γ as well as those which belong to Δ may be complex. It seems that abductive problems expressed by syntactically complex abductive questions are not easy to solve. In order to obey the Erotetic Decomposition Principle, the first step in solving an abductive problem (or, to put it differently, in answering an abductive question) is to make this problem 'simpler'. In the formulation of erotetic rules of inference we make use of the α, β-notation[3]. These rules constitute an erotetic calculus for CPL[4]. We denote it by the symbol $\mathbb{E}^{\mathsf{CPL}}$.

A sequent of a premise question distinguished in the scheme of a rule of $\mathbb{E}^{\mathsf{CPL}}$ is called *premise sequent* and sequent(s) distinguished in the conclusion is (are) called a *conclusion sequent(s)* of a given rule. In a similar manner we can define the *premise formulas* and *conclusion formulas* of a given rule (and of a given sequent). Occasionally we will say that a conclusion formula *results from* a premise formula.

Sequences of questions governed by erotetic rules of inference are *Socratic transformations.*

Definition 4 (Socratic transformation). *A finite sequence of questions* $\boldsymbol{s} = \langle s_1, \ldots, s_n \rangle$ *is a* Socratic transformation *(s-transformation) of the question* $?(\Phi)$ *by means of* $\mathbb{E}^{\mathsf{CPL}}$ *iff the following conditions hold:*

1. $s_1 = ?(\Phi)$.
2. s_i *results from* s_{i-1} *(where* $i > 1$*) by an application of a rule of* $\mathbb{E}^{\mathsf{CPL}}$.

An s-transformation $\mathbf{s} = \langle s_1, \ldots, s_n \rangle$ is said to be *complete* iff the last term of $\mathbf{s}$, s_n, is a minimal question. A sequent ϕ is *basic* if ϕ is of one of the following forms: $\Gamma, B, \Gamma' \vdash \Delta, B, \Delta'$ or $\Gamma, B, \Gamma', \neg B, \Gamma'' \vdash \Delta$ or $\Gamma \vdash \Delta, B, \Delta', \neg B, \Delta''$. Naturally, each basic sequent is closed.

[3] α, β-notation was introduced by Smullyan in [17] to simplify metalogical considerations.

[4] A version of this calculus was introduced by Wiśniewski in [21]. In his approach only one formula can occur in the consequent of the sequent.

Definition 5 (Socratic proof). *A Socratic proof (s-proof) of a sequent $\Gamma \vdash \Delta$ in $\mathbb{E}^{\mathsf{CPL}}$ is a finite s-transformation $\boldsymbol{s}$ of the question $?(\Gamma \vdash \Delta)$, such that each constituent of the last question of $\boldsymbol{s}$ is a basic sequent.*

Socratic transformation of a question $?(\Gamma \vdash \Delta)$ is *successful* iff there exists a socratic proof of $\Gamma \vdash \Delta$. In the light of definition 3 there are no successful s-transformations of abductive questions.

2 How to answer an abductive question

To answer an abductive question $?(\Gamma \vdash \Delta)$ we employ the following procedure:

Step 1. Create a complete s-transformation of the question $?(\Gamma \vdash \Delta)$; the last question of this s-transformation is based on a sequence of sequents each of which consists of literals only.

Step 2. Apply some abductive rules (to be introduced later on) to this last question; each rule is *local* in the sense that only one sequent at a time is active in such a rule.

Step 3. Combine the results of the applications of rules using a conjunction; the resulting hypothesis has the form $H = A_1 \wedge \ldots \wedge A_n$, where each A_i $(1 \leq i \leq n)$ is the conclusion of an abductive rule.

There are several criteria of evaluation of abductive hypotheses [1,10]. To implement those criteria we need some auxiliary notions which allow us to illustrate the proposed method. Let $?(\Gamma \vdash \Delta)$ be an abductive question and H — an answer to the initial question. An abductive hypothesis has the following form $H = A_1 \wedge \ldots \wedge A_n$ where A_i $(1 \leq i \leq n)$ is a formula which closes some sequent in the last question of an s-transformation of $?(\Gamma \vdash \Delta)$.

We distinguish the following criteria[5]: 1. *Consistency:* $\Gamma \cup \{H\}$ is consistent. 2. *Significance:* $H \nvdash_{\mathsf{CPL}} \Delta$.

2.1 Abductive rules

In this section our aim is to design rules for answering abductive questions which produce hypotheses/answers, which are significant and consistent with the knowledge base.

Definition 6 (Partial answer). *Let $Q = ?(\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n)$ be an abductive question. Let us further assume that the sequent $\Gamma_i \vdash \Delta_i$ (for some i, where $1 \leq i \leq n$) is open.* Partial answer *for Q is such a formula A that the addition of A to the Γ_i results in $\Gamma_i \vdash \Delta_i$ becoming a closed sequent or a sequent which after transformation to the atomic sequent is also a closed one.*

[5] Similar constraints are also defined in [1, p. 74] (Aliseda describes those two criteria as constituting the *consistent* and the *explanatory* Abductive Explanatory Styles respectively) and as properties of the abduction for Abductive Logic Programming in [5].

Table 2. Examples of abductive rules

$$\frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta''\ ;\ \Psi)}{\bar{l}}\ \mathbf{R}^1_{abd} \qquad \frac{?(\Phi\ ;\ \Theta, l, \Theta' \vdash \Theta'', k, \Theta'''\ ;\ \Psi)}{l \to k}\ \mathbf{R}^2_{abd}$$

Definition 7 (Abductive rule). *Let Q be a minimal abductive question and A be a partial answer for Q. The premise of an* abductive rule *is Q and the conclusion is A.*

In this paper we propose two rules for answering abductive questions 2. Note that the premises of abductive rules are questions (minimal abductive questions) and conclusions are declarative formulas. Thus, abductive rules enable a kind of inference between a question and an answer to that question. Note also that abductive rules close an active sequent (a sequent distinguished in the premise of a rule) in a natural way: atomic sequent is closed either by making its antecedent contradictory, or by making some connection between antecedent and consequent.

2.2 Restrictions for abductive rules

The proposed rules cannot be applied without some restrictions, if we want to maintain the consistency or significance of generated abductive hypotheses. To state those restrictions precisely we need some auxiliary notions, which are familiar from the work of Hintikka and Fitting (see for example [6]).

Definition 8 (Downward saturated set). *Let Γ be a sequence of formulas of $\mathcal{L}^*_{\mathsf{CPL}}$. By a* downward saturated set *(or* Hintikka set*) corresponding to a sequence Γ we mean a set $\mathfrak{U}_\Gamma$, which fulfils the following conditions:*

- *(i) if A is a term of Γ, then $A \in \mathfrak{U}_\Gamma$,*
- *(ii) if $\alpha \in \mathfrak{U}_\Gamma$, then $\alpha_1 \in \mathfrak{U}_\Gamma$ and $\alpha_2 \in \mathfrak{U}_\Gamma$,*
- *(iii) if $\beta \in \mathfrak{U}_\Gamma$, then $\beta_1 \in \mathfrak{U}_\Gamma$ or $\beta_2 \in \mathfrak{U}_\Gamma$,*
- *(iv) if $\neg\neg A \in \mathfrak{U}_\Gamma$, then $A \in \mathfrak{U}_\Gamma$.*
- *(v) nothing more belongs to $\mathfrak{U}_\Gamma$ except those formulas which enter $\mathfrak{U}_\Gamma$ on the grounds of conditions (i)–(iv).*

A Hintikka set $\mathfrak{U}_\Gamma$ is satisfied under a Boolean valuation v (or is consistent) iff each element of $\mathfrak{U}_\Gamma$ is true under v. A Hintikka set $\mathfrak{U}_\Gamma$ is inconsistent iff for every v, at least one formula in $\mathfrak{U}_\Gamma$ is false under v. If $\mathfrak{U}_\Gamma = \emptyset$, then $\mathfrak{U}_\Gamma$ is satisfied by each Boolean valuation ($\mathfrak{U}_\Gamma$ is valid).

Definition 9 (Consistency property). *By a* consistency property *corresponding to a sequence Γ we mean a finite set $\mathfrak{U}^c_\Gamma = \{\mathfrak{U}^1_\Gamma, \ldots, \mathfrak{U}^n_\Gamma\}$, which contains all Hintikka sets for Γ that do not contain complementary literals.*

Lemma 1. *If a non-empty sequence of formulas Γ is satisfiable, then at least one downward saturated set corresponding to Γ belongs to consistency property of Γ.*

Lemma 2 (Hintikka's Lemma). *For arbitrary Γ, each set belonging to the consistency property of Γ is satisfiable.*

Corollary 1. *A Hintikka set $\mathfrak{U}_\Gamma$ is inconsistent iff for some literal l, $l \in \mathfrak{U}_\Gamma$ and $\bar{l} \in \mathfrak{U}_\Gamma$.*

Definition 10 (Dual downward saturated set). *Let Δ be a sequence of formulas of $\mathcal{L}^*_{\mathsf{CPL}}$. By a* dual downward saturated set *(or* dual Hintikka set*) corresponding to a sequence Δ we mean a set $\mathfrak{W}_\Delta$, which fulfils the following conditions:*

(i) if A is a term of Δ, then $A \in \mathfrak{W}_\Delta$,
(ii) if $\alpha \in \mathfrak{W}_\Delta$, then $\alpha_1 \in \mathfrak{W}_\Delta$ or $\alpha_2 \in \mathfrak{W}_\Delta$,
(iii) if $\beta \in \mathfrak{W}_\Delta$, then $\beta_1 \in \mathfrak{W}_\Delta$ and $\beta_2 \in \mathfrak{W}_\Delta$,
(iv) if $\neg\neg A \in \mathfrak{W}_\Delta$, then $A \in \mathfrak{W}_\Delta$.
(v) nothing more belongs to $\mathfrak{W}_\Delta$ except those formulas which enter $\mathfrak{W}_\Delta$ on the grounds of conditions (i)–(iv).

A dual Hintikka set $\mathfrak{W}_\Delta$ is d-satisfied under a Boolean valuation v iff at least one element of $\mathfrak{W}_\Delta$ is true under v. A dual Hintikka set $\mathfrak{W}_\Delta$ is d-satisfied by each classical valuation ($\mathfrak{W}_\Gamma$ is valid) iff there is no Boolean valuation v such that each formula in $\mathfrak{W}_\Delta$ is false under v. If $\mathfrak{W}_\Delta = \emptyset$, then $\mathfrak{W}_\Delta$ is d-inconsistent.

Corollary 2. *A dual Hintikka set $\mathfrak{W}_\Delta$ is d-satisfied by each classical valuation ($\mathfrak{W}_\Delta$ is d-valid) iff for some l, $l \in \mathfrak{W}_\Delta$ and $\bar{l} \in \mathfrak{W}_\Delta$.*

Definition 11 (Non-validity property). *By a* non-validity property *corresponding to a sequence Δ we mean a finite set $\mathfrak{W}^{nv}_\Delta = \{\mathfrak{W}^1_\Delta, \ldots, \mathfrak{W}^n_\Delta\}$, which contains all dual Hintikka sets for Δ that do not contain complementary literals.*

Lemma 3 (Dual Hintikka's Lemma). *For an arbitrary Δ, each set belonging to the non-validity property of Δ is not d-valid.*

Let us consider the rules from Definition 2 again. Let us focus on rule $\mathbf{R}^1_{abd}$ and let $?(\Gamma \vdash \Delta)$ be the first question of a complete s-transformation which ends with the minimal question of the following form $?(\Phi \ ; \ \Theta, l, \Theta' \vdash \Theta'' \ ; \ \Psi)$. We have two kinds of restrictions which guarantee the consistency and significance of abductive hypotheses generated by $\mathbf{R}^1_{abd}$.

Restriction 1 (Consistency restriction on $\mathbf{R}^1_{abd}$) *There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $l \notin \mathfrak{U}_\Gamma$.*

Restriction 2 (Significance restriction on $\mathbf{R}^1_{abd}$) *There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $\bar{l} \notin \mathfrak{W}_\Delta$.*

Let us focus on rule $\mathbf{R}^2_{abd}$ and let $?(\Gamma \vdash \Delta)$ be the first question of a complete s-transformation which ends with a minimal question of the following form $?(\Phi \ ; \ \Theta, l, \Theta' \vdash \Theta'' \ ; \ \Psi)$. We have two kinds of restrictions which guarantee the consistency and significance of our abductive hypotheses generated by $\mathbf{R}^2_{abd}$.

Restriction 3 (Consistency restriction on $\mathbf{R}^2_{abd}$) *There exists a set $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ such that $l \notin \mathfrak{U}_\Gamma$ or $\overline{k} \notin \mathfrak{U}_\Gamma$.*

Restriction 4 (Significance restriction on $\mathbf{R}^2_{abd}$) *There exists a set $\mathfrak{W}_\Delta \in \mathfrak{W}^{nv}_\Delta$ such that $\bar{l} \notin \mathfrak{W}_\Delta$ or $k \notin \mathfrak{W}_\Delta$.*

Let $\mathbf{s} = \langle Q_1, \ldots, Q_n \rangle$ be a complete s-transformation of the question $?(\Gamma \vdash A)$, ϕ be an active sequent of an abductive rule, l, k–active literals and $\mathfrak{U}^c_\Gamma$ the consistency property for Γ. The application of a given rule to ϕ with restriction, with respect to l (or l and k), generates a set of Hintikka sets which are not compatible with a given restriction. Let us call this set $\mathfrak{U}^{c-}_\Gamma$. Now we can define a new set $\mathfrak{U}^{c+}_\Gamma = \mathfrak{U}^c_\Gamma \setminus \mathfrak{U}^{c-}_\Gamma$, which is a consistency property compatible with a given partial answer. If $\mathfrak{U}^{c+}_\Gamma$ results from $\mathfrak{U}^c_\Gamma$ in the case of an application of rule $\mathbf{R}^1_{abd}$ with an active literal l, then by $\mathfrak{U}^{c+\bar{l}}_\Gamma$ we mean the consistency property such that literal $\bar{l}$ is added to each element of $\mathfrak{U}^{c+}_\Gamma$. In the case of an application of $\mathbf{R}^2_{abd}$, the consistency property $\mathfrak{U}^{c+\bar{l},k}_\Gamma$ is the effect of adding literals $\bar{l}$ and k to each element of $\mathfrak{U}^{c+}_\Gamma$.

In a similar manner, we can define the set $\mathfrak{W}^{nv+}_\Delta = \mathfrak{W}^{nv}_\Delta \setminus \mathfrak{W}^{nv-}_\Gamma$ and the set $\mathfrak{W}^{nv+l}_\Gamma$. In the case of an application of $\mathbf{R}^2_{abd}$ things are slightly more complicated. First we have to construct two sets $\mathfrak{W}^{nv+l}_\Delta$ and $\mathfrak{W}^{nv+\overline{k}}_\Delta$, and then the set $\mathfrak{W}^{nv+l,\overline{k}}_\Delta = \mathfrak{W}^{nv+l}_\Delta \cup \mathfrak{W}^{nv+\overline{k}}_\Delta$.

In order to prove a correctness of the procedure we need to modify its second step.

Step 2*. Apply some abductive rules to the last question with consistency (significance) restriction; after each application of an abductive rule modify the consistency property (non-validity property) in order to make it compatible with a given partial answer.

In the example at the end of the paper we show in details how the procedure works. Before that we introduce the following lemmas (sometimes without proofs, if they are trivial) and theorems proving our method to be correct.

Lemma 4. *Let $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ be a downward saturated set corresponding to some Γ. If a literal $l \notin \mathfrak{U}_\Gamma$, then the set $\mathfrak{U}_\Gamma \cup \{\bar{l}\}$ is consistent.*

Proof. $\mathfrak{U}_\Gamma$ is consistent by definition of the consistency property. Let us assume that $l \notin \mathfrak{U}_\Gamma$. If $\mathfrak{U}_\Gamma \cup \{\bar{l}\}$ is inconsistent then $l \in \mathfrak{U}_\Gamma \cup \{\bar{l}\}$, which contradicts the assumption. □

Lemma 5. *Let $\mathfrak{U}_\Gamma \in \mathfrak{U}^c_\Gamma$ be a downward saturated set corresponding to some Γ. If $l \notin \mathfrak{U}_\Gamma$ or $\overline{k} \notin \mathfrak{U}_\Gamma$, then the set $\mathfrak{U}_\Gamma \cup \{l \to k\}$ is consistent.*

Theorem 1. *Each abductive hypothesis generated by the procedure, where each abductive rule is applied with a consistency restriction is consistent with the initial knowledge base.*

Proof. The proof follows from Lemma 4, Lemma 5 and from the construction of $\mathfrak{U}^{c+}_{\Gamma}$. □

Lemma 6. *Let $\mathfrak{W}_{\Delta} \in \mathfrak{W}^{nv}_{\Delta}$ be a dual downward saturated set corresponding to some Δ. If a literal $\bar{l} \notin \mathfrak{W}_{\Delta}$, then the set $\mathfrak{W}_{\Delta} \cup \{l\}$ is not valid.*

Proof. We know that $\mathfrak{W}_{\Delta}$ is not valid, i.e. there exists a valuation v such that each formula in $\mathfrak{W}_{\Delta}$ is false under v. Since $\bar{l} \notin \mathfrak{W}_{\Delta}$ we can assume that $v(l) = 0$. It follows that $\mathfrak{W}_{\Delta}$ is not valid. □

Lemma 7. *Let $\mathfrak{W}_{\Delta} \in \mathfrak{W}^{nv}_{\Delta}$ be a dual downward saturated set corresponding to some Δ. If $\bar{l} \notin \mathfrak{W}_{\Delta}$ or $k \notin \mathfrak{W}_{\Delta}$, then the set $\mathfrak{W}_{\Delta} \cup \{l\}$ is not valid or $\mathfrak{W}_{\Delta} \cup \{\bar{k}\}$ is not valid.*

Lemma 8. *$l \nvdash_{\mathsf{CPL}} A_1 \vee \ldots \vee A_n$ (where each A_i $(1 \leq i \leq n)$ is a literal) if and only if a dual Hintikka set $\mathfrak{W} = \{\bar{l}, A_1, \ldots, A_n\}$ is not valid.*

Proof. $(\rightarrow)$ Assume that $l \nvdash_{\mathsf{CPL}} A_1 \vee \ldots \vee A_n$. There exists a classical valuation v such that $v(l) = 1$ and $v(A_1 \vee \ldots \vee A_n) = 0$. In this case $v(\bar{l}) = 0$ and each formula in $\mathfrak{W}$ is false under v. Therefore $\mathfrak{W}$ is not valid.
$(\leftarrow)$ Assume $\mathfrak{W}$ is not valid. There exists a classical valuation v, such that each formula in $\mathfrak{W}$ is false under v. In this case $v(l) = 1$ and $v(A_1 \vee \ldots \vee A_n) = 0$. Therefore $l \nvdash_{\mathsf{CPL}} A_1 \vee \ldots \vee A_n$. □

Lemma 9. *$l \rightarrow k \nvdash_{\mathsf{CPL}} A_1 \vee \ldots \vee A_n$ (where each A_i $(1 \leq i \leq n)$ is a literal) if and only if a dual Hintikka set $\mathfrak{W} = \{l, A_1, \ldots, A_n\}$ is not valid or $\mathfrak{W} = \{\bar{k}, A_1, \ldots, A_n\}$ is not valid.*

Theorem 2. *Each abductive hypothesis generated by the procedure, where each abductive rule is applied with a significance restriction, is significant.*

Proof. The proof is a consequence of Lemma 6, Lemma 7 and the construction of $\mathfrak{W}^{nv+}_{\Gamma}$. □

Theorem 3. *Each abductive hypothesis generated by the procedure, where each abductive rule is applied with a significance and consistency restriction is significant and consistent.*

Proof. The proof follows from Theorem 2 and Theorem 1. □

Let us consider the following example. The knowledge base $\Gamma = \langle p \rightarrow (z \rightarrow q), r \wedge s \rangle$, and $\Delta = \langle r \rightarrow q \rangle$. Therefore the initial question is of the following form: $?(p \rightarrow (z \rightarrow q), r \wedge s \vdash r \rightarrow q)$, and the last question of s-transformation is of the form: $?(\neg p, r, s \vdash \neg r, q \ ; \ \neg z, r, s \vdash \neg r, q \ ; \ q, r, s \vdash \neg r, q)$.

After constructing the s-transformation of our problem we have to calculate Hintikka and dual Hintikka sets. From the knowledge base $\Gamma = \langle p \rightarrow (z \rightarrow q), r \wedge s \rangle$ we can generate the following seven Hintikka sets and the consistency property ($\mathfrak{U}^{c}_{\Gamma} = \{\mathfrak{U}^{1}_{\Gamma}, \mathfrak{U}^{2}_{\Gamma}, \mathfrak{U}^{3}_{\Gamma}, \mathfrak{U}^{4}_{\Gamma}, \mathfrak{U}^{5}_{\Gamma}, \mathfrak{U}^{6}_{\Gamma}, \mathfrak{U}^{7}_{\Gamma}\}$):

$\mathfrak{U}^{1}_{\Gamma} = \{p \rightarrow (z \rightarrow q), r \wedge s, r, s, \neg p, z \rightarrow q, \neg z, q\}$

$\mathfrak{U}_\Gamma^2 = \{p \to (z \to q), r \wedge s, r, s, \neg p, z \to q, \neg z\}$
$\mathfrak{U}_\Gamma^3 = \{p \to (z \to q), r \wedge s, r, s, \neg p, z \to q, q\}$
$\mathfrak{U}_\Gamma^4 = \{p \to (z \to q), r \wedge s, r, s, \neg p\}$
$\mathfrak{U}_\Gamma^5 = \{p \to (z \to q), r \wedge s, r, s, z \to q, \neg z, q\}$
$\mathfrak{U}_\Gamma^6 = \{p \to (z \to q), r \wedge s, r, s, z \to q, \neg z\}$
$\mathfrak{U}_\Gamma^7 = \{p \to (z \to q), r \wedge s, r, s, z \to q, q\}$

The abductive goal is $\Delta = \langle r \to q \rangle$ and we can generate the following dual Hintikka set: $\mathfrak{W}_\Delta^1 = \{r \to q, \neg r, q\}$ and non-validity property: $\mathfrak{W}_\Delta^{nv} = \{\mathfrak{W}_\Delta^1\}$

Now, let us consider the set of possible abductive hypotheses generated by the introduced rules. The first open sequent $\neg p, r, s \vdash \neg r, q$ can be closed by formulas which belong to the set $\Sigma_1 \cup \Sigma_2$, where:

1. $\Sigma_1 = \{p, \neg r, \neg s\}$ is the set of formulas generated by means of the application of the rule $\mathbf{R}_{abd}^1$;
2. $\Sigma_2 = \{\neg p \to \neg r, r \to \neg r, s \to \neg r, \neg p \to q, r \to q, s \to q\}$ is the set of formulas generated by means of the application of the rule $\mathbf{R}_{abd}^2$.

The second open sequent $\neg z, r, s \vdash \neg r, q$ can be closed by formulas which belong to the set $\Sigma_1^* \cup \Sigma_2^*$, where:

1*. $\Sigma_1^* = \{z, \neg r, \neg s\}$ is the set of formulas generated by means of the application of the rule $\mathbf{R}_{abd}^1$;
2*. $\Sigma_2^* = \{\neg z \to \neg r, r \to \neg r, s \to \neg r, \neg z \to q, r \to q, s \to q\}$ is the set of formulas generated by means of the application of the rule $\mathbf{R}_{abd}^2$.

An abductive answer to the initial question $?(p \to (z \to q), r \wedge s \vdash r \to q)$ is a conjunction of formulas which close all the open sequents in the minimal abductive question (conjunction of all partial answers). In this particular case the set of all answers has the form:

$$\{A \wedge B \mid A \in \Sigma_i,\ B \in \Sigma_i^*, \text{ for } i \in \{1, 2\}\}$$

Note that not all of these answers would be generated, when abductive rules are used along with restrictions. Let us look at some examples of answers. Some of them will be consistent and significant while other will not.

(a) $H = p \wedge z$. In this case rule $\mathbf{R}_{abd}^1$ has been applied to close both open sequents. Formula p closes the first open sequent. Moreover the consistency restriction is fulfilled: there exists a set $\mathfrak{U}_\Gamma^i \in \mathfrak{U}_\Gamma^c$ such that $\neg p \notin \mathfrak{U}_\Gamma$. In fact there are three such sets: $\mathfrak{U}_\Gamma^5$, $\mathfrak{U}_\Gamma^6$, $\mathfrak{U}_\Gamma^7$. Formula z closes the second open sequent. The consistency restriction is also fulfilled in this case, because there exists a set $\mathfrak{U}_\Gamma^i \in \mathfrak{U}_\Gamma^{c+p}$, namely $\mathfrak{U}_\Gamma^{7+p}$, such that $\neg z \notin \mathfrak{U}_\Gamma^{7+p}$. Thus H is consistent with the knowledge base. This hypothesis is also significant. Significance restriction is fulfilled in the case of partial answer p, because there exists a set $\mathfrak{W}_\Delta^i \in \mathfrak{W}_\Delta^{nv}$ such that $p \notin \mathfrak{W}_\Delta^i$, namely $\mathfrak{W}_\Delta^1$. The significance restriction is also fulfilled for the second partial answer z, because $z \notin \mathfrak{W}_\Delta^{1+p}$.

(b) $H = p \wedge (r \to \neg r)$. This hypothesis is significant because of similar reasons as in the previous example (a). However it is not consistent with the knowledge base due to the fact that for each $\mathfrak{U}^{i+p}_{\Gamma} \in \mathfrak{U}^{c+p}_{\Gamma}$, $r \in \mathfrak{U}^{i+p}_{\Gamma}$, which contradicts consistency restriction on $\mathbf{R}^2_{abd}$.
(c) $H = \neg r$. In this case rule $\mathbf{R}^1_{abd}$ has been applied to close both open sequents. This hypothesis is neither consistent nor significant, because each $\mathfrak{U}^{i}_{\Gamma} \in \mathfrak{U}^{c}_{\Gamma}$ is such that $r \in \mathfrak{U}_{\Gamma}$ and each $\mathfrak{W}^{i}_{\Delta} \in \mathfrak{W}^{nv}_{\Delta}$ is such that $\neg r \in \mathfrak{W}^{i}_{\Delta}$.

3 Summary and further work

In this article we introduced Abductive Question-Answer System for classical propositional logic. We interpret the abductive problem as an abductive question: *what should be added to the knowledge base Γ in order to be able to derive a fact φ?*, where φ is not derivable from Γ. Firstly, (possibly) complex initial abductive question is decomposed into a minimal abductive question. Afterwards, partial answers are generated for the minimal abductive question. The abductive hypothesis is obtained by combining all partial answers with conjunction. If partial answers are generated along with restrictions, obtained abductive hypothesis have 'desired' properties i.e., it is consistent with the knowledge base Γ and φ is not obtainable from the abductive hypothesis alone. Therefore, AQAS integrates generation and evaluation of abductive hypotheses.

The knowledge-based systems are better and more often described by means of modal or paraconsistent logics. Therefore, our future work is concerned with the application of the Abductive Question-Answer System for these logics. Our future work will also cover the implementation of the Abductive Question-Answer System in programming language. This will enable us to test the system on huge datasets and compare it with solutions that already exist, such as the one presented by Komosiński [10] or those proposed on the ground of Abductive Logic Programming [5].

Acknowledgements

This work has been supported by the Polish National Science Center, grant no. 2012/04/A/HS1/00715 (first author) and DEC-2013/10/E/HS1/00172 (second author).

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