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An Abductive Question-Answer System for the Minimal Logic of Formal Inconsistency mbC

Abstract. The aim in this paper is to define an Abductive Question-Answer System for the minimal logic of formal inconsistency mbC. As a proof-theoretical basis we employ the Socratic proofs method. The system produces abductive hypotheses; these are answers to abductive questions concerning derivability of formulas from sets of formulas. We integrated the generation of and the evaluation of hypotheses via constraints of consistency and significance being imposed on the system rules.

Keywords: Abduction, Inferential erotetic logic, Minimal logic of formal inconsistency.

1. Introduction

In abductive reasoning we aim at filling a certain gap between a knowledge base Γ and a puzzling phenomenon A , unattainable from Γ (cf. [15, 23]). The commonly accepted schema is this: from an observation A and the known rule if H , then A , infer H (see [21, 5.189]). However, this schema may be and has been studied in detail in different ways, which leads to different models of abduction [12, 24]. From a computational point of view, it is of particular appeal to consider an algorithmic perspective, according to which an abductive hypothesis H “is legitimately dischargeable to the extent to which it makes it possible to prove (or compute) from a database, a formula not provable (or computable) from it, as it is currently structured” [13, p. 88]. A brief overview of procedures for generation of abductive hypotheses defined in such a spirit can be found in [17].

There are four primary ingredients of the algorithmic account of abduction [17, p. 2]: (i) a basic logic (which determines the language of specification of A , H and Γ), (ii) a proof method (which determines the exact mechanics of the procedure of generation of abducibles), (iii) a hypotheses generation mechanism (which determines the way the chosen proof method is applied

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in order to generate abducibles), and (iv) an implementation of criteria for comparative evaluation of different abducibles.

In this paper we seek to introduce a framework for automated generation and evaluation of abductive hypotheses in the form of an Abductive Question-Answer System. Our proof-theoretical basis is set up by the Socratic proofs method [26]: we shall employ the concepts of an erotetic calculus and of a Socratic transformation of a question [28], which is proven to be effective in an automated proof search [9, 18, 27, 29]. As the reader will see, there are close affinities between erotetic calculi and sequent calculi with semantically reversible rules [28, p. 98]. The basic logic of choice is the minimal logic of formal inconsistency mbC , which is one of the logics of formal inconsistency [5, 6]. On the one hand, these logics, being paraconsistent, are tools for reasoning under conditions which do not presuppose consistency [20, p. 465]. On the other hand, they “have a remarkable way of reintroducing consistency into the non-classical picture: they internalize the very notions of consistency and inconsistency at the object-language level.” [6, p. 1]. Therefore, one can reconstruct classical logic *inside* paraconsistent logic. Moreover, more expressive language allows for formulation of such abductive hypotheses that could not be obtained by means of classical logic. In light of this arguments we can say that the abductive procedure we describe in this paper does not contradict procedures based on classical logic but extends them.

We define an Abductive Question-Answer System for mbC in the form of an erotetic calculus augmented with *abductive rules*, which allow for systematic search for answers to *abductive questions*, that is, *abductive hypotheses*. Consistency and significance constraints, imposed on abductive rules applications, warrants that generated abductive hypotheses meet those basic proof-theoretical criteria. Thus in the case of our system comparative evaluation of hypotheses is embedded into the procedure for their generation. This is worth noticing, as separating hypotheses generation and evaluation is by far more popular approach within the algorithmic perspective (cf. [13, p. 47], [17]).

We start with introducing the minimal logic of formal inconsistency, mbC (Section 2) and its proof theory employing the Socratic proofs method (Section 3). On this basis we define our abductive procedure (Section 4), including an algorithm for generation of abductive hypotheses (p. 24).

2. Minimal Logic of Formal Inconsistency

Logics of formal inconsistency (LFIs) are paraconsistent logics, which are able to remodel classically valid reasoning, by means of some special operator \circ . A formula of the form $\circ A$ should be read *it is consistent that A* or *A behaves classically*. The main intuition is the following: when we reason with the set of premises $\Gamma = \{A_1, \dots, A_n\}$ such that there is no A_i ($1 \leq i \leq n$) which contains consistency operator \circ , we use deductive machinery of some paraconsistent logic. But when we obtain an information that some of these formulas are consistent, i.e. the \circ operator occurs somewhere in Γ , we can use much stronger deductive machinery of classical logic to reason classically about fragments of Γ . Suppose we have the following set of premises $\{p, \sim p\}$, where \sim is a paraconsistent negation. Using some paraconsistent logic, like CLuN [2] for example, one cannot deduce an arbitrary formula from this set. But if we have an additional premise which says that p is *safe* or *consistent*, formalized by the formula $\circ p$, we can deduce an arbitrary formula B , just like in classical logic.

We use the language \mathcal{L}_{mbC} of minimal logic of formal inconsistency, which consists of a countably infinite set $Var = \{p_1, p_2, \dots\}$ of propositional variables and \neg (classical negation), \sim (paraconsistent negation), \circ (consistency operator), \wedge (conjunction), \vee (disjunction), \rightarrow (implication) as primitive connectives. The set of well-formed formulas (wffs for short) is defined as usual:

$$A, B ::= p_i \mid \neg A \mid \sim A \mid \circ A \mid A \wedge B \mid A \vee B \mid A \rightarrow B$$

By a *literal* we mean a propositional variable or its negation. Literals are denoted by l, k, m and so on. If $l = p_i$, then \bar{l} means $\neg p_i$ and if $l = \neg p_i$, then $\bar{l} = p_i$. The literals l and \bar{l} are called *complementary literals*. Naturally, $l = \bar{\bar{l}}$.

The Hilbert-style system for mbC consists of the following axioms and rules [6]:

¹Although classical negation of a formula A , $\neg A$, can be defined in mbC by $A \rightarrow \perp$, where $\perp =_{df} p \wedge (\sim p \wedge \circ p)$, we prefer to introduce classical negation as a primitive connective.

- | | |
|---|--|
| (1) $A \rightarrow (B \rightarrow A)$ | (2) $(A \rightarrow B) \rightarrow (((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$ |
| (3) $A \rightarrow (B \rightarrow (A \wedge B))$ | (4) $(A \wedge B) \rightarrow A$ |
| (5) $(A \wedge B) \rightarrow B$ | (6) $A \rightarrow (A \vee B)$ |
| (7) $B \rightarrow (A \vee B)$ | (8) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$ |
| (9) $A \vee (A \rightarrow B)$ | (10) $A \vee \neg A$ |
| (11) $A \rightarrow (\neg A \rightarrow B)$ | (12) $A \vee \sim A$ |
| (13) $\circ A \rightarrow (A \rightarrow (\sim A \rightarrow B))$ | |
- (MP) If $\vdash_{\text{mbC}} A$ and $\vdash_{\text{mbC}} A \rightarrow B$, then $\vdash_{\text{mbC}} B$

DEFINITION 1. (*mbC-semivaluation*)² An *mbC-semivaluation* is a function $v : \text{FOR}^{\text{mbC}} \rightarrow \{0, 1\}$ which behaves in a standard way in the case of classical connectives and the following conditions are satisfied:

- (\sim) if $v(\sim A) = 0$, then $v(A) = 1$;
- (\circ) if $v(\circ A) = 1$, then either $v(A) = 0$ or $v(\sim A) = 0$.

From a proof-theoretical point of view it is more convenient to work with the notion of *mbC-valuation* instead of *semivaluation* for the following reason. The notion of *mbC-semivaluation* has a drawback: the truth value of some formulas is not determined by its subformulas (if $v(A) = 1$ then $v(\sim A)$ is not determined). In order to assign values to formulas of the form $\sim A$ and $\circ A$ in a consistent manner, we introduce a new assignment function $\bar{\lambda}$. Using the notion of *mbC-valuation* we are able to give a simple soundness and completeness theorems for our proof method.

DEFINITION 2. (*mbC-valuation*)³ Let $\text{FOR}_{\text{mbC}}^{\sim \circ}$ denote the set of all formulas of FOR_{mbC} of the form ' $\sim A$ ' or ' $\circ A$ '. A function $\lambda : \text{FOR}_{\text{mbC}} \rightarrow \{0, 1\}$ is *mbC-valuation* iff λ behaves classically in the case of classical connectives and there exists an assignment function $\bar{\lambda} : \text{FOR}_{\text{mbC}}^{\sim \circ} \rightarrow \{0, 1\}$ such that the following conditions hold:

- (1) $\lambda(\sim A) = 1$ iff $\lambda(A) = 0$ or $\bar{\lambda}(\sim A) = 1$;
- (2) $\lambda(\circ A) = 1$ iff $(\lambda(A) = 0$ and $\bar{\lambda}(\circ A) = 1)$ or $(\lambda(\sim A) = 0$ and $\bar{\lambda}(\circ A) = 1)$

For further reference let us state truth conditions for formulas of the form ' $\neg \sim A$ ' and ' $\neg \circ A$ ':

- (1*) $\lambda(\sim A) = 0$ iff $\lambda(A) = 1$ and $\bar{\lambda}(\sim A) = 0$;
- (2*) $\lambda(\circ A) = 0$ iff $\lambda(A) = \lambda(\sim A) = 1$ or $\bar{\lambda}(\circ A) = 0$.

²The notion of *mbC-semivaluation* is described in [8] and in [6] (but under the name "bivaluation semantics for *mbC*"). The term *semivaluation* is sometimes used in a slightly different sense, see for example [19].

³The notion of *mbC-valuation* is defined in [8].

Now we define a new language $\mathcal{L}_{\text{mbC}}^+$ which is an extension of the language \mathcal{L}_{mbC} . For proof-theoretical reasons we add to the latter the symbol χ . The set of well formed formulas of $\mathcal{L}_{\text{mbC}}^+$ is defined in the following way:

$$\text{FOR}_{\text{mbC}}^+ = \text{FOR}_{\text{mbC}} \cup \{\chi A : A \in \text{FOR}_{\text{mbC}}^{\sim \circ}\} \cup \{\neg \chi A : A \in \text{FOR}_{\text{mbC}}^{\sim \circ}\}$$

For the extended language $\mathcal{L}_{\text{mbC}}^+$ we define an extension of **mbC**-valuation. Semantically, the introduced operator, χ , does not change the truth value of a given formula. This operator is only a syntactic device for identifying formulas preceded by consistency operator and paraconsistent negation, i.e. formulas whose truth value is not determined by its subformulas.

DEFINITION 3. (*mbC-valuation for $\mathcal{L}_{\text{mbC}}^+$*) An *extended mbC-valuation* is a function $\lambda^\# : \text{FOR}_{\text{mbC}}^+ \longrightarrow \{0, 1\}$, which behaves in the same manner on the set FOR_{mbC} as λ and the following conditions are satisfied:

- (1) $\lambda^\#(\chi \sim A) = \lambda(\sim A)$;
- (2) $\lambda^\#(\neg \chi \sim A) = \lambda(\neg \sim A)$;
- (3) $\lambda^\#(\chi \circ A) = \lambda(\circ A)$;
- (4) $\lambda^\#(\neg \chi \circ A) = \lambda(\neg \circ A)$.

Note that in the extended language $\mathcal{L}_{\text{mbC}}^+$ we are able to express normal forms of formulas of the language \mathcal{L}_{mbC} . This fact has the following consequences: it is relatively easy to design *invertible* sequent calculus rules for non-classical connectives and the completeness theorem for the introduced calculus can be based on the strategy of counter-model construction. These advantages will be apparent in the next section on proof theory of **mbC**.

3. Proof Theory of **mbC**

There are several proof-theoretical descriptions of the logic **mbC**: there is a standard tableau method for several LFIs (where rules operate on signed formulas) [4], **KE** tableau method (a variant of standard tableau method, in which some form of the cut rule is essential), which is also implemented [20]. Moreover there is a sequent calculus for **mbC** [10] and a system based on resolution rule and grounded in Inferential Erotetic Logic [8]. Our approach is based on Inferential Erotetic Logic as well and we use some techniques and concepts introduced in [8], but we have to forgo the resolution rule in order to obtain a simple and intuitive model of abductive reasoning. In fact, our system is akin to some version of hypersequent calculus [27].

The language $\mathcal{L}_{\text{mbC}}^2$ is an object-level language in which our erotetic calculi will be worded. The meaningful expressions of the language $\mathcal{L}_{\text{mbC}}^2$ belong to two disjoint sets. The first one consists of *declarative well-formed formulas* (d-wffs for short). The second one is a set of *erotetic well-formed formulas* (e-wffs or simply *questions*).

To obtain the vocabulary of $\mathcal{L}_{\text{mbC}}^2$ we add the following signs: \vdash (turnstile which intuitively stands for a derivability relation in **mbC**), $?$ (a question mark for constructing questions of $\mathcal{L}_{\text{mbC}}^2$) and $,$ (a comma), $;$ (a semicolon), to the vocabulary of $\mathcal{L}_{\text{mbC}}^+$.

DEFINITION 4. Let Γ, Δ be finite, non-empty, sequences of formulas of $\mathcal{L}_{\text{mbC}}^+$. An *atomic declarative formula* or *sequent* of $\mathcal{L}_{\text{mbC}}^2$ is of the following form:

$$\Gamma \vdash \Delta$$

DEFINITION 5. Questions of $\mathcal{L}_{\text{mbC}}^2$ have the following form:

$$?(\Phi)$$

where Φ is a finite, non-empty sequence of sequents of $\mathcal{L}_{\text{mbC}}^2$.

Let $\Phi = \langle \Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n \rangle$ be a sequence of sequents of $\mathcal{L}_{\text{mbC}}^2$. The question

$$?(\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n)$$

is interpreted as follows:

Is it the case that Δ_1 is entailed by Γ_1 and ... and Δ_n is entailed by Γ_n ?

Terms of Φ are called *constituents* of the question $?(\Phi)$.

If Γ entails in **mbC** Δ then we say that the sequent $\Gamma \vdash \Delta$ of $\mathcal{L}_{\text{mbC}}^2$ is *closed*, otherwise it is *open*.

A sequent ϕ is *basic* iff ϕ is of one of the following forms ($\Gamma, \Gamma', \Gamma'', \Delta, \Delta', \Delta''$ may be empty):

- $\Gamma, B, \Gamma' \vdash \Delta, B, \Delta'$,
- $\Gamma, B, \Gamma', \neg B, \Gamma'' \vdash \Theta$,
- $\Gamma, \neg B, \Gamma', B, \Gamma'' \vdash \Theta$,
- $\Theta \vdash \Delta, B, \Delta', \neg B, \Delta''$,
- $\Theta \vdash \Delta, \neg B, \Delta', B, \Delta''$,

where $B \in \text{FOR}_{\text{mbC}}^+$. Naturally, each basic sequent is closed.

The erotetic calculus \mathbb{E}^{mbC} for the logic **mbC** consists of the rules for classical connectives and specific **mbC** rules.

Table 1. α/β —Formulas

α	α_1	α_2	β	β_1	β_2
$A \wedge B$	A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B
$\neg(A \rightarrow B)$	A	$\neg B$	$A \rightarrow B$	$\neg A$	B

Table 2. Rules for classical connectives

$\frac{?(\Phi; \Gamma, \alpha, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, \alpha_1, \alpha_2, \Gamma' \vdash \Delta; \Psi)} \mathbf{L}_\alpha$	$\frac{?(\Phi; \Gamma \vdash \Delta, \alpha, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, \alpha_1, \Delta'; \Gamma \vdash \Delta, \alpha_2, \Delta'; \Psi)} \mathbf{R}_\alpha$
$\frac{?(\Phi; \Gamma, \beta, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, \beta_1, \Gamma' \vdash \Delta; \Gamma, \beta_2, \Gamma' \vdash \Delta; \Psi)} \mathbf{L}_\beta$	$\frac{?(\Phi; \Gamma \vdash \Delta, \beta, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, \beta_1, \beta_2, \Delta'; \Psi)} \mathbf{R}_\beta$
$\frac{?(\Phi; \Gamma, \neg\neg A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, A, \Gamma' \vdash \Delta; \Psi)} \mathbf{L}_{\neg\neg}$	$\frac{?(\Phi; \Gamma \vdash \Delta, \neg\neg A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, A, \Delta'; \Psi)} \mathbf{R}_{\neg\neg}$

Table 3. Specific rules of mbC

$\frac{?(\Phi; \Gamma, \sim A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, \neg A, \Gamma' \vdash \Delta; \Gamma, \chi \sim A, \Gamma' \vdash \Delta : \Psi)} \mathbf{L}_{\sim}$
$\frac{?(\Phi; \Gamma \vdash \Delta, \sim A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, \neg A, \chi \sim A, \Delta'; \Psi)} \mathbf{R}_{\sim}$
$\frac{?(\Phi; \Gamma, \neg \sim A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, A, \neg \chi \sim A, \Gamma' \vdash \Delta; \Psi)} \mathbf{L}_{\neg \sim}$
$\frac{?(\Phi; \Gamma \vdash \Delta, \neg \sim A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, A, \Delta'; \Gamma \vdash \Delta, \neg \chi \sim A, \Delta'; \Psi)} \mathbf{R}_{\neg \sim}$
$\frac{?(\Phi; \Gamma, \circ A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, \neg A, \chi \circ A, \Gamma' \vdash \Delta; \Gamma, \neg \sim A, \chi \circ A, \Gamma' \vdash \Delta; \Psi)} \mathbf{L}_\circ$
$\frac{?(\Phi; \Gamma \vdash \Delta, \circ A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, \chi \circ A, \Delta'; \Gamma \vdash \Delta, \neg A, \neg \sim A, \Delta'; \Psi)} \mathbf{R}_\circ$
$\frac{?(\Phi; \Gamma, \neg \circ A, \Gamma' \vdash \Delta; \Psi)}{?(\Phi; \Gamma, A, \sim A, \Gamma' \vdash \Delta; \Gamma, \neg \chi \circ A, \Gamma' \vdash \Delta; \Psi)} \mathbf{L}_{\neg \circ}$
$\frac{?(\Phi; \Gamma \vdash \Delta, \neg \circ A, \Delta'; \Psi)}{?(\Phi; \Gamma \vdash \Delta, A, \neg \chi \circ A, \Delta'; \Gamma \vdash \Delta, \sim A, \neg \chi \circ A, \Delta'; \Psi)} \mathbf{R}_{\neg \circ}$

DEFINITION 6. (*Socratic transformation*) A finite sequence of questions $\mathbf{s} = \langle s_1, \dots, s_n \rangle$ is a *Socratic transformation* (s-transformation) of a question $?(\Phi)$ by means of \mathbb{E}^{mbC} iff the following conditions hold:

1. $s_1 = ?(\Phi)$.
2. s_i results from s_{i-1} (where $i > 1$) by an application of a rule of \mathbb{E}^{mbC} .

DEFINITION 7. (*Socratic proof*) A Socratic proof (s-proof) of a sequent $\Gamma \vdash \Delta$ in \mathbb{E}^{mbC} is a finite s-transformation \mathbf{s} of the question $?(\Gamma \vdash \Delta)$, such that each constituent of the last question of \mathbf{s} is a basic sequent.

An *active sequent* is specified in a premise of a given \mathbb{E}^{mbC} rule scheme. A *principal sequent* is specified in a conclusion of a given \mathbb{E}^{mbC} rule scheme. For each \mathbb{E}^{mbC} rule scheme exactly one sequent is active in the premise of that rule. A formula which is specified in an active sequent of a given rule scheme is called an *active formula* of that rule scheme. Formula(s) which is (are) specified in the principal sequent of a given rule is (are) called *principal formula(s)* of that rule.

DEFINITION 8. (*unanalyzable formulas*) The formulas of one of the following forms: (i) p_i ; (ii) $\neg p_i$; (iii) $\chi \sim A$; (iv) $\neg \chi \sim A$; (v) $\chi \circ A$; (vi) $\neg \chi \circ A$, and only them, are *mbC-unanalyzable*.

unanalyzable formulas are those that are not active formulas in any premise of any \mathbb{E}^{mbC} rule. This means that these formulas cannot be further decomposed by means of introduced rules; however they guarantee that the rules for non-classical connectives are sound and invertible (see LEMMA 2).

If a sequent consists of unanalyzable formulas only, it is called an *atomic sequent*. If each constituent of a question Q is atomic, then Q is called an *atomic question*. An s-transformation \mathbf{s} is called *complete* iff the last term of \mathbf{s} is an atomic question. There are two kinds of unanalyzable formulas: *positive unanalyzable* formulas are of the form (i), (iii), (v); *negative unanalyzable* formulas are of the form (ii), (iv), (vi). *Complementary unanalyzable formulas* are pairs of formulas: (i) and (ii), (iii) and (iv), (v) and (vi), where the negative unanalyzable formula is obtainable by addition of the classical negation to the positive unanalyzable formula, and similarly, the positive unanalyzable formula is obtainable by removal of the classical negation from the beginning of the negative unanalyzable formula.

We will say that the sequent $\Gamma \vdash \Delta$ is *valid* iff there is no valuation $\lambda^\#$ such that $\lambda^\#(A) = 1$ for every term A of Γ and $\lambda^\#(B) = 0$ for every term B of Δ .

DEFINITION 9. (*complexity*) The complexity of a formula $A \in \text{FOR}_{\text{mbC}}^+$, $\text{com}(A)$ is defined inductively as follows:

1. $\text{com}(A) = 0$, where $A \in \text{Var}$;
2. $\text{com}(\chi \sim A) = \text{com}(\chi \circ A) = 0$;
3. $\text{com}(\neg A) = \text{com}(A) + 1$;
4. $\text{com}(\sim A) = \text{com}(A) + 2$;
5. $\text{com}(\circ A) = \text{com}(A) + 3$;
6. $\text{com}(A \otimes B) = \text{com}(A) + \text{com}(B) + 1$, where $\otimes \in \{\wedge, \vee, \rightarrow\}$.

This concept of formula complexity is not standard because it does not measure the number of occurrences of propositional connectives (such as items 2, 4, and 5) in the formula. At the semantic level, although connectives \sim and \circ differ from the other ones by being slightly more complicated (their truth value is not always determined by the truth values of their subformulas), the introduced notion of complexity reflects this difference. Note that if A is an unanalyzable formula then $\text{com}(A) \leq 1$.

COROLLARY 3.1. (*reduction*) For rule schemes $\mathbf{L}_\alpha, \mathbf{R}_\alpha, \mathbf{L}_\beta, \mathbf{L}_\beta, \mathbf{R}_\sim, \mathbf{R}_\sim, \mathbf{L}_{\neg\sim}, \mathbf{R}_{\neg\sim}$, $\text{com}(B) < \text{com}(A)$ and $\text{com}(C) < \text{com}(A)$, where A is an active formula of that rule and B and C are principal formulas of that rule. For rule schemes $\mathbf{L}_\circ, \mathbf{R}_\circ, \mathbf{L}_{\neg\circ}, \mathbf{R}_{\neg\circ}$, $\text{com}(B) < \text{com}(A)$, $\text{com}(C) < \text{com}(A)$, $\text{com}(D) < \text{com}(A)$, where B, C, D are principal formulas of that rule and A is an active formula of that rule.

Note that the above corollary follows directly from the definition of the measures of complexity.

LEMMA 1. (existence of atomic questions) Let $\Gamma \vdash \Delta$ be an arbitrary sequent of $\mathcal{L}_{\text{mbC}}^?$. There exists a finite s -transformation $\mathbf{s} = \langle s_1, \dots, s_n \rangle$, where $s_1 = ?(\Gamma \vdash \Delta)$ and s_n is an atomic question.

PROOF. If $\Gamma \vdash \Delta$ is a sequent containing only unanalyzable formulas (atomic sequent), then $s_1 = ?(\Gamma \vdash \Delta)$ is an atomic question and $\mathbf{s} = \langle s_1 \rangle$ is finite s -transformation.

If the sequent $\Gamma \vdash \Delta$ is not atomic, then from LEMMA 3.1 we know that the rules which could be applied to $?(\Gamma \vdash \Delta)$ reduce the complexity of some formula in Γ or Δ . As both Γ and Δ are finite, by applying rules of \mathbb{E}^{mbC} consecutively we obtain an atomic question. ■

LEMMA 2. For every \mathbb{E}^{mbC} rule scheme the active sequent of that rule is valid iff the principal sequent(s) is (are) valid.

PROOF. Proof goes by cases. Let us consider \mathbf{R}_\circ rule scheme. Let us assume that the sequent $\Gamma \vdash \Delta, \circ A, \Delta'$ is valid. The only non-trivial case is when $\lambda^\#(\circ A) = 1$. By the DEFINITION 3 we have that ($\lambda^\#(A) = 0$ and $\lambda^\#(\chi \circ A) = 1$) or ($\lambda^\#(\sim A) = 0$ and $\lambda^\#(\chi \circ A) = 1$). What follows, ($\lambda^\#(A) = 0$ or $\lambda^\#(\sim A) = 0$) and $\lambda^\#(\chi \circ A) = 1$. Principal sequents of the \mathbf{R}_\circ rule scheme are: $\Gamma \vdash \Delta, \chi \circ A, \Delta'$ and $\Gamma \vdash \Delta, \neg A, \neg \sim A, \Delta'$. For the first principal sequent we have that $\lambda^\#(\chi \circ A) = 1$ and for the second, $\lambda^\#(\neg A) = 1$ or $\lambda^\#(\neg \sim A) = 1$. Similarly, it is easy to see (in the light of DEFINITION 3) that the validity of the principal sequents ensures the validity of the active sequent of \mathbf{R}_\circ rule scheme.

The proof goes analogically for the rest of \mathbb{E}^{mbC} rule schemes. ■

LEMMA 3. (countermodel) *Let $\Gamma \vdash \Delta$ be an atomic and not basic sequent. There exists an extended mbC-valuation $\lambda^\#$ such that $\lambda^\#(A) = 1$, for each term A of Γ and $\lambda^\#(B) = 0$, for each term B of Δ .*

PROOF. Note that the following clauses define an assignment which determines an extended mbC-valuation which invalidates $\Gamma \vdash \Delta$.

1. if D is a term of Γ (where $D \in \text{Var}$), then $\lambda^\#(D) = 1$;
2. if $\neg D$ is a term of Γ (where $D \in \text{Var}$), then $\lambda^\#(D) = 0$;
3. if $\chi \sim D$ is a term of Γ , then $\lambda^\#(\sim D) = 1$;
4. if $\neg \chi \sim D$ is a term of Γ , then $\lambda^\#(\sim D) = 0$;
5. if $\chi \circ D$ is a term of Γ , then $\lambda^\#(\circ D) = 1$;
6. if $\neg \chi \circ D$ is a term of Γ , then $\lambda^\#(\circ D) = 0$;
7. if D is a term of Δ ($D \in \text{Var}$), then $\lambda^\#(D) = 0$;
8. if $\neg D$ is a term of Δ ($D \in \text{Var}$), then $\lambda^\#(D) = 1$;
9. if $\chi \sim D$ is a term of Δ , then $\lambda^\#(\sim D) = 0$;
10. if $\neg \chi \sim D$ is a term of Δ , then $\lambda^\#(\sim D) = 1$;
11. if $\chi \circ D$ is a term of Δ , then $\lambda^\#(\circ D) = 0$;
12. if $\neg \chi \circ D$ is a term of Δ , then $\lambda^\#(\circ D) = 1$.
13. if D is not a term of Γ nor Δ , then $\lambda^\#(D) = 0$.

We will show that this assignment determines an extended mbC-valuation, i.e. it is not the case that there is a formula A such that $\lambda^\#(A) = 1$ and $\lambda^\#(A) = 0$. As is easily seen, it cannot happen when A is unanalyzable formula. Assume it cannot happen in the case of formulas B and C and it happens in the case of $B \wedge C$, i.e. (i) $\lambda^\#(B \wedge C) = 1$ and (ii) $\lambda^\#(B \wedge C) = 0$.

From (i) it follows that $\lambda^\#(B) = 1$ and $\lambda^\#(C) = 1$, and from (ii) it follows that $\lambda^\#(B) = 0$ or $\lambda^\#(C) = 0$. In this case $\lambda^\#(B) = 1$ and $\lambda^\#(B) = 0$ or $\lambda^\#(C) = 1$ and $\lambda^\#(C) = 0$, contrary to our assumption. A similar argument applies in the case of other connectives. ■

THEOREM 1. (completeness) *If a sequent ϕ is valid, then ϕ is provable in \mathbb{E}^{mbC} .*

PROOF. Assume that ϕ is valid but not provable in \mathbb{E}^{mbC} . Therefore, by **LEMMA 1** there exists a finite Socratic transformation $\mathbf{s} = \langle s_1, \dots, s_n \rangle$ of the question $?(\phi)$ such that at least one constituent ψ of the last question of \mathbf{s} is an atomic but not basic sequent. By **LEMMA 3** there exists an extended **mbC**-valuation, which invalidates ψ . Thus at least one constituent of the last question s_n of \mathbf{s} is not valid. Therefore, by **LEMMA 2** we know that at least one constituent of the question s_{n-1} is not valid. By applying **LEMMA 2** consecutively we arrive at the conclusion that the sequent ϕ is not valid, contrary to the assumption. ■

4. Abduction in **mbC**

Following the algorithmic account of abduction, we interpret the abductive problem as a requirement to fill the deductive gap between the premises and the conclusion. In our Abductive Question-Answer System, such requests will be expressed through abductive questions.

DEFINITION 10. (*Abductive question*) An *abductive question* (or *abductive problem*) has the following form:

$$?(\Psi)$$

where Ψ is a non-empty sequence of sequents such that at least one term of Ψ is an open sequent of $\mathcal{L}_{\text{mbC}}^?$.

- If $\Psi = \langle \phi \rangle$ is an one-term sequence, then the question $?(\Psi)$ is called *simple abductive question*.
- If $\Psi = \langle \phi_1, \dots, \phi_n \rangle$ and for each i ($1 \leq i \leq n$), ϕ_i is an atomic sequent, then the question $?(\Psi)$ is called a *minimal abductive question*.

An atomic abductive question:

$$?(\Gamma \vdash \Delta) \tag{1}$$

can be read as follows:

$$\text{Which formulas close } \ulcorner \Gamma \vdash \Delta \urcorner? \quad (2)$$

A compound abductive question

$$?(\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n) \quad (3)$$

can be read as follows:

$$\text{Which formulas close every term of } \langle \Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n \rangle? \quad (4)$$

Let $V_{\Gamma_i \vdash \Delta_i} = \{p_1^i, \dots, p_{n_i}^i\}$ be the set of all propositional variables occurring in wffs in sequences Γ_i or Δ_i from (3), then $V_\Phi = \bigcup_{i=1}^n (V_{\Gamma_i \vdash \Delta_i})$ is the set of all propositional variables occurring in wffs in sequences from Φ , where $\Phi = \langle \Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n \rangle$.

DEFINITION 11. (*Direct analytic answers*) A wff B is a *direct analytic answer* to the abductive question $?(\Phi)$ if and only if each propositional variable, which occurs in B belongs to V_Φ .

DEFINITION 12. (*Correct analytic answers*) A wff B is a *correct analytic answer* to the abductive question $?(\Phi)$ if and only if

- B is a direct analytic answer to $?(\Phi)$,
- B closes each open term of Φ .

The fact that B is a correct analytic answer for a question Q will be denoted as: $B \in c(Q)$.

As an example let us consider the question:

$$Q^* = ?(p, \neg(q \wedge r), \neg s \vdash z ; r \rightarrow s, \neg s, p \vdash r) \quad (5)$$

- the set of all propositional variables occurring in Q^* : $V_{Q^*} = \{p, q, r, s, z\}$,
- exemplary direct analytic answers to Q^* : $p, q, p \rightarrow q$,
- exemplary correct analytic answers to Q^* : $\neg p, s, \neg p \wedge z$.

THEOREM 4.1. (Invertibility of the rules of \mathbb{E}^{mbC}) *Let $\mathbf{s} = \langle Q_1, \dots, Q_n \rangle$ be a complete Socratic transformation of the question Q_1 in \mathbb{E}^{mbC} .*

★ *For some wff C , if $C \in c(Q_n)$, then $C \in c(Q_1)$.*

PROOF. Proof goes by induction on the length of the s-transformation \mathbf{s} and by an inspection of the rules of \mathbb{E}^{mbC} .

The case of $\text{length}(\mathbf{s}) = 1$ is trivial.⁴

⁴ $\text{Length}(\mathbf{s})$ denotes the number of elements in sequence \mathbf{s} .

Table 4. Examples of abductive rules

$\frac{?(\Phi ; \Theta, l, \Theta' \vdash \Theta'' ; \Psi)}{\bar{l}} \mathbf{R}_{abd}^1$	$\frac{?(\Phi ; \Theta, l, \Theta' \vdash \Theta'', k, \Theta''' ; \Psi)}{l \rightarrow k} \mathbf{R}_{abd}^2$
$\frac{?(\Phi ; \Theta, \chi \circ p, \Theta', \chi \sim p, \Theta'' \vdash \Theta''' ; \Psi)}{p} \mathbf{R}_{abd}^3$	$\frac{?(\Phi ; \Theta, \chi \sim p, \Theta', \chi \circ p, \Theta'' \vdash \Theta''' ; \Psi)}{p} \mathbf{R}_{abd}^{3*}$
$\frac{?(\Phi ; \Theta, \chi \circ p, \Theta', p, \Theta'' \vdash \Theta''' ; \Psi)}{\sim p} \mathbf{R}_{abd}^4$	$\frac{?(\Phi ; \Theta, p, \Theta', \chi \circ p, \Theta'' \vdash \Theta''' ; \Psi)}{\sim p} \mathbf{R}_{abd}^{4*}$
$\frac{?(\Phi ; \Theta, p, \Theta', \chi \sim p, \Theta'' \vdash \Theta''' ; \Psi)}{\circ p} \mathbf{R}_{abd}^5$	$\frac{?(\Phi ; \Theta, \chi \sim p, \Theta', p, \Theta'' \vdash \Theta''' ; \Psi)}{\circ p} \mathbf{R}_{abd}^{5*}$

Assume that the property holds for the arbitrary s-transformation \mathbf{s} , where $length(\mathbf{s}) = i$. We will show that it is also valid for the s-transformation \mathbf{s}' , where $length(\mathbf{s}') = i+1$. Let us also assume that the last rule applied in \mathbf{s}' was \mathbf{L}_α . Let formula C close each constituent of the last question of \mathbf{s}' (Q_{i+1}). Let $\Gamma_1, \alpha_1, \alpha_2, \Gamma_2 \vdash \Delta$ be the conclusion sequent of \mathbf{L}_α . We know from assumption that C closes this sequent, i.e. $C, \Gamma_1, \alpha_1, \alpha_2, \Gamma_2 \vdash_{mbc} \bigvee \Delta$. Since $v(\alpha) = 1$ iff $v(\alpha_1) = 1$ and $v(\alpha_2) = 1$ the formula C closes the premise sequent of the last applied rule, namely $C, \Gamma_1, \alpha, \Gamma_2 \vdash_{mbc} \bigvee \Delta$. The length of the s-transformation with the deleted last question equals i . A similar argument applies for the other rules. ■

DEFINITION 13. (*Partial answer*) Let $Q = ?(\Gamma_1 \vdash \Delta_1, \dots, \Gamma_n \vdash \Delta_n)$ be an abductive question. Let us further assume that the sequent $\Gamma_i \vdash \Delta_i$ (for some i , where $1 \leq i \leq n$) is open. *Partial answer* for Q is such a formula A that the addition of A to the Γ_i results in $\Gamma_i \vdash \Delta_i$ becoming a closed sequent or a sequent which after transformation to the atomic sequent is also a closed one.

Note that it could be the case that a partial answer is also a correct analytic answer, i.e. it closes all open sequents of Q .

DEFINITION 14. (*Abductive rule*) Let Q be a minimal abductive question and A be a partial answer for Q . The premise of an *abductive rule* is Q and the conclusion is A .

Each rule in Table 4 has a question as the premise and a declarative formula as a conclusion (which is a partial answer to the question-premise). A formula, which is a conclusion, when added to the antecedent of an active sequent in an abductive rule, makes this antecedent inconsistent (in the classical sense) or generates a link between antecedent and succedent.

Most abductive procedures consist of two steps: abductive hypotheses generation and subsequent evaluation of generated hypotheses against predefined criteria [17]. There are several candidates for such criteria [1, 8]. In this paper we are concerned with just two of them, which are of fundamental importance from the proof-theoretical point of view.

The first one is *consistency*: an abductive hypothesis should be consistent with the initial data or knowledge base. Note that the notion of consistency used here is relative to the logic mbC i.e. abductive hypothesis of the form $\sim p$ is consistent with the knowledge base of the form $\Gamma = \{p, q, r\}$, but not with the knowledge base $\Gamma' = \{p, q, \circ p\}$ for example. The reason for introducing the consistency criterion is that we do not want a knowledge base to become inconsistent or *trivial* in the sense that every formula could be inferred from it. Consistency criterion may be also called *non-triviality* criterion.

The second one is *significance*: an abductive hypothesis should not allow to derive Δ by itself, that is, in filling this deductive gap both abductive hypothesis and the initial database should be significant.

Another point is that we do not want to carry out two separate steps for the generation and evaluation of the generated abductive hypothesis, but rather to build an abductive procedure that generates the *good* abductive hypotheses. Therefore, we will not implement the evaluation criteria of the abductive hypotheses, but we will implement rules that can make the construction of the abductive hypotheses consistent and / or significant. These rules are implemented as restrictions placed on the application of the rules.

Abductive rules from Table 4 generate partial answers that can be inconsistent with the knowledge base Γ or it could be the case, that such a generated partial answer is itself sufficient to derive Δ . As we would like to rule out such possibilities, we introduce *downward* (or Hintikka) and *dual downward saturated sets*. Partial answers that are generated along with the restriction concerning downward saturated set (we will call it *consistency constraints*) will be consistent with the knowledge base Γ . Similarly, partial answers that are generated with the restriction concerning dual downward saturated set (we will call it *significance constraints*) will not be too strong, i.e. Δ would not be obtainable from the partial answer alone.

DEFINITION 15. (*Downward saturated set*) Let Γ be a sequence of formulas of $\mathcal{L}_{\text{mbC}}^+$. By a *downward saturated set* (or *Hintikka set*) corresponding to a sequence Γ we mean a set \mathfrak{U}_Γ , which fulfils the following conditions:

- (i) if A is a term of Γ , then $A \in \mathfrak{U}_\Gamma$,
- (ii) if $\alpha \in \mathfrak{U}_\Gamma$, then $\alpha_1 \in \mathfrak{U}_\Gamma$ and $\alpha_2 \in \mathfrak{U}_\Gamma$,

- (iii) if $\beta \in \mathfrak{U}_\Gamma$, then $\beta_1 \in \mathfrak{U}_\Gamma$ or $\beta_2 \in \mathfrak{U}_\Gamma$,
- (iv) if $\neg\neg A \in \mathfrak{U}_\Gamma$, then $A \in \mathfrak{U}_\Gamma$,
- (v) if $\sim A \in \mathfrak{U}_\Gamma$, then $\neg A \in \mathfrak{U}_\Gamma$ or $\chi \sim A \in \mathfrak{U}_\Gamma$,
- (vi) if $\neg \sim A \in \mathfrak{U}_\Gamma$, then $A \in \mathfrak{U}_\Gamma$ and $\neg\chi \sim A \in \mathfrak{U}_\Gamma$,
- (vii) if $\circ A \in \mathfrak{U}_\Gamma$, then $(\neg A \in \mathfrak{U}_\Gamma$ and $\chi \circ A \in \mathfrak{U}_\Gamma)$ or $(\neg \sim A \in \mathfrak{U}_\Gamma$ and $\chi \circ A \in \mathfrak{U}_\Gamma)$,
- (viii) if $\neg \circ A \in \mathfrak{U}_\Gamma$, then $(A \in \mathfrak{U}_\Gamma$ and $\sim A \in \mathfrak{U}_\Gamma)$ or $\neg\chi \circ A \in \mathfrak{U}_\Gamma$,
- (ix) nothing more belongs to \mathfrak{U}_Γ except those formulas which enter \mathfrak{U}_Γ on the grounds of conditions (i)–(viii).

A Hintikka set \mathfrak{U}_Γ is satisfied under extended mbC-valuation $\lambda^\#$ (or is consistent) iff each element of \mathfrak{U}_Γ is true under v . A Hintikka set \mathfrak{U}_Γ is inconsistent iff for every $\lambda^\#$, at least one formula in \mathfrak{U}_Γ is false under $\lambda^\#$. If $\mathfrak{U}_\Gamma = \emptyset$, then \mathfrak{U}_Γ is satisfied by each extended mbC-valuation (\mathfrak{U}_Γ is valid).

DEFINITION 16. (*Consistency property*) By a *consistency property* corresponding to a sequence Γ we mean a finite set $\mathfrak{U}_\Gamma^c = \{\mathfrak{U}_\Gamma^1, \dots, \mathfrak{U}_\Gamma^n\}$, which contains all Hintikka sets for Γ that does not contain complementary unanalyzable formulas.

Let us recall Hintikka’s well-known result:

LEMMA 4. (Hintikka’s Lemma) *For arbitrary Γ , each set belonging to the consistency property of Γ is satisfiable.*

The idea in the proof is to construct a valuation which sends each unanalyzable formula to 1. The next step is to show by induction that it can be extended to satisfy all formulas from Γ . For a detailed proof see for example [11].

COROLLARY 4.2. *A Hintikka set \mathfrak{U}_Γ is inconsistent iff for some unanalyzable formula A , $A \in \mathfrak{U}_\Gamma$ and $\neg A \in \mathfrak{U}_\Gamma$.*

LEMMA 5. *If a non-empty sequence of formulas Γ is satisfiable, then at least one downward saturated set corresponding to Γ belongs to consistency property of Γ .*

PROOF. Assume no downward saturated set corresponding to Γ belongs to consistency property of Γ . Thus all such sets contain complementary unanalyzable formulas. Due to the fact that the construction of downward saturated sets reflects mbC-valuation, Γ cannot be satisfiable. ■

The notion of *downward saturated set* can be dualized in order to tackle the problem of significance restriction. Detailed study of such sets in the context of First-Order Logic can be found in [7].

DEFINITION 17. (*Dual downward saturated set*) Let Δ be a sequence of formulas of $\mathcal{L}_{\text{mbC}}^+$. By a *dual downward saturated set* (or *dual Hintikka set*) corresponding to a sequence Δ we mean a set \mathfrak{W}_Δ , which fulfils the following conditions:

- (i) if A is a term of Δ , then $A \in \mathfrak{W}_\Delta$,
- (ii) if $\alpha \in \mathfrak{W}_\Delta$, then $\alpha_1 \in \mathfrak{W}_\Delta$ or $\alpha_2 \in \mathfrak{W}_\Delta$,
- (iii) if $\beta \in \mathfrak{W}_\Delta$, then $\beta_1 \in \mathfrak{W}_\Delta$ and $\beta_2 \in \mathfrak{W}_\Delta$,
- (iv) if $\neg\neg A \in \mathfrak{W}_\Delta$, then $A \in \mathfrak{W}_\Delta$,
- (v) if $\sim A \in \mathfrak{W}_\Delta$, then $\neg A \in \mathfrak{W}_\Delta$ and $\chi \sim A \in \mathfrak{W}_\Delta$,
- (vi) if $\neg \sim A \in \mathfrak{W}_\Delta$, then $A \in \mathfrak{W}_\Delta$ or $\neg \chi \sim A \in \mathfrak{W}_\Delta$,
- (vii) if $\circ A \in \mathfrak{W}_\Delta$, then $\chi \circ A \in \mathfrak{W}_\Delta$ or $(\neg A \in \mathfrak{W}_\Delta$ and $\neg \sim A \in \mathfrak{W}_\Delta)$,
- (viii) if $\neg \circ A \in \mathfrak{W}_\Delta$, then $(A \in \mathfrak{W}_\Delta$ and $\neg \chi \circ A \in \mathfrak{W}_\Delta)$ or $(\sim A \in \mathfrak{W}_\Delta$ and $\neg \chi \circ A \in \mathfrak{W}_\Delta)$,
- (ix) nothing more belongs to \mathfrak{W}_Δ except those formulas which enter \mathfrak{W}_Δ on the grounds of conditions (i)–(viii).

A dual Hintikka set \mathfrak{W}_Δ is *d-satisfied* under extended mbC-valuation $\lambda^\#$ iff at least one element of \mathfrak{W}_Δ is true under $\lambda^\#$. A dual Hintikka set \mathfrak{W}_Δ is *d-satisfied* by each extended mbC-valuation (\mathfrak{W}_Δ is *d-valid*) iff there is no extended mbC-valuation $\lambda^\#$ such that each formula in \mathfrak{W}_Δ is false under $\lambda^\#$. If $\mathfrak{W}_\Delta = \emptyset$, then \mathfrak{W}_Δ is *d-inconsistent*.

COROLLARY 4.3. *A dual Hintikka set \mathfrak{W}_Δ is d-satisfied by each extended mbC-valuation (\mathfrak{W}_Δ is d-valid) iff for some unanalyzable formula A , $A \in \mathfrak{W}_\Delta$ and $\neg A \in \mathfrak{W}_\Delta$.*

DEFINITION 18. (*Non-validity property*) By a *non-validity property* corresponding to a sequence Δ we mean a finite set $\mathfrak{W}_\Delta^{nv} = \{\mathfrak{W}_\Delta^1, \dots, \mathfrak{W}_\Delta^n\}$, which contains all dual Hintikka sets for Δ that do not contain complementary unanalyzable formulas.

LEMMA 6. (*Dual Hintikka's Lemma*) *For an arbitrary Δ , each set belonging to the non-validity property of Δ is not d-valid.*

The idea of the proof of dual Hintikka's lemma is analogous to that of Hintikka's lemma. But now we want to construct a valuation which sends

each unanalyzable formula to 0. Then we can easily extend such valuation to falsify all formulas from Δ .

Table 5 contains constraints to rules from Table 4. The intuitions underlying those constraints are the following: in cases when we want to generate abductive hypothesis that is consistent with the knowledge base Γ , we look for those formulas, that are consistent with at least one Hintikka set. In other words, we can say that we are looking for a formula that is true under some mbC-valuation $\lambda^\#$, under which all formulas from Γ are also true.

Constraints for significance of the abductive hypothesis reflect similar notions. We are looking for those hypotheses that do not make at least one Hintikka set d-valid. In other words, when we extend such dual Hintikka set by the negation of a formula which is our hypothesis,⁵ and turn such extended dual Hintikka set into a formula by linking all elements from this set by disjunctions, we do not want to obtain a formula which is true under every mbC-valuation $\lambda^\#$.

LEMMA 7. *Let $\mathfrak{U}_\Gamma \in \mathfrak{U}_\Gamma^c$ be a downward saturated set corresponding to some Γ . If an unanalyzable formula $l \notin \mathfrak{U}_\Gamma$, then the set $\mathfrak{U}_\Gamma \cup \{\bar{l}\}$ is consistent.*

PROOF. \mathfrak{U}_Γ is consistent by definition of the consistency property. Let us assume that $l \notin \mathfrak{U}_\Gamma$. If $\mathfrak{U}_\Gamma \cup \{\bar{l}\}$ is inconsistent then $l \in \mathfrak{U}_\Gamma \cup \{\bar{l}\}$, which contradicts the assumption. ■

LEMMA 8. *Let $\mathfrak{U}_\Gamma \in \mathfrak{U}_\Gamma^c$ be a downward saturated set corresponding to some Γ . If $l \notin \mathfrak{U}_\Gamma$ or $\bar{k} \notin \mathfrak{U}_\Gamma$, then the set $\mathfrak{U}_\Gamma \cup \{l \rightarrow k\}$ is consistent.*

PROOF. The proof is analogous to the proof of LEMMA 7. ■

Now we are going to prove that abductive hypotheses generated by abductive rules used in accordance with the constraints are consistent with the initial knowledge base and significant. For that reason we introduce an algorithm 1 (see page 24) which creates s-transformation, consistency and significance properties for an initial question $Q = ?(\Gamma \vdash \Delta)$, and then uses abductive rules along with restrictions to produce abductive hypotheses.

THEOREM 4.4. *Each abductive hypothesis generated by Algorithm 1, where each abductive rule is applied with a consistency constraint is consistent with the initial knowledge base.*

PROOF. The proof follows from LEMMAS 7 and 8 and from the construction of \mathfrak{U}_Γ^{c+} . ■

⁵Addition of the negation simulates the transfer of the formula from the antecedent to the consequent of a sequent. Abductive hypotheses are always added to the knowledge base, which is situated in the antecedent of a sequent.

Table 5. Consistency and significance restrictions for abductive rules

Abductive rule	Consistency restriction: <i>There exists a set $\mathcal{M}_\Gamma \in \mathcal{M}_\Gamma^c$ such that...</i>	Significance restriction: <i>There exists a set $\mathcal{W}_\Delta \in \mathcal{W}_\Delta^{mv}$ such that...</i>
R_{abd}^1	$l \notin \mathcal{M}_\Gamma$	$\bar{l} \notin \mathcal{W}_\Delta$
R_{abd}^2	$l \notin \mathcal{M}_\Gamma$ or $\bar{k} \notin \mathcal{M}_\Gamma$	$\bar{l} \notin \mathcal{W}_\Delta$ or $\bar{k} \notin \mathcal{W}_\Delta$
R_{abd}^3 (R_{abd}^{3*})	$op \notin \mathcal{M}_\Gamma$ or $\sim p \notin \mathcal{M}_\Gamma$	$p \notin \mathcal{W}_\Delta$
R_{abd}^4 (R_{abd}^{4*})	$op \notin \mathcal{M}_\Gamma$ or $p \notin \mathcal{M}_\Gamma$	$\sim p \notin \mathcal{W}_\Delta$
R_{abd}^5 (R_{abd}^{5*})	$p \notin \mathcal{M}_\Gamma$ or $\sim p \notin \mathcal{M}_\Gamma$	$op \notin \mathcal{W}_\Delta$

LEMMA 9. Let $\mathfrak{W}_\Delta \in \mathfrak{W}_\Delta^{nv}$ be a dual downward saturated set corresponding to some Δ . If an unanalyzable formula $\bar{l} \notin \mathfrak{W}_\Delta$, then the set $\mathfrak{W}_\Delta \cup \{l\}$ is not valid.

PROOF. We know that \mathfrak{W}_Δ is not valid, i.e. there exists an extended mbC-valuation $\lambda^\#$ such that each formula in \mathfrak{W}_Δ is false under $\lambda^\#$. Since $\bar{l} \notin \mathfrak{W}_\Delta$ we can assume that $\lambda^\#(l) = 0$. It follows that \mathfrak{W}_Δ is not valid. ■

LEMMA 10. Let $\mathfrak{W}_\Delta \in \mathfrak{W}_\Delta^{nv}$ be a dual downward saturated set corresponding to some Δ . If $\bar{l} \notin \mathfrak{W}_\Delta$ or $k \notin \mathfrak{W}_\Delta$, then the set $\mathfrak{W}_\Delta \cup \{l\}$ is not valid or $\mathfrak{W}_\Delta \cup \{\bar{k}\}$ is not valid.

PROOF. The proof is analogous to the proof of LEMMA 9. ■

LEMMA 11. $l \not\vdash_{\text{mbC}} A_1 \vee \dots \vee A_n$ (where each A_i ($1 \leq i \leq n$) is a literal) if and only if a dual Hintikka set $\mathfrak{W} = \{\bar{l}, A_1, \dots, A_n\}$ is not valid.

PROOF. (\rightarrow) Assume that $l \not\vdash_{\text{mbC}} A_1 \vee \dots \vee A_n$. There exists an extended mbC-valuation $\lambda^\#$ such that $\lambda^\#(l) = 1$ and $\lambda^\#(A_1 \vee \dots \vee A_n) = 0$. In this case $\lambda^\#(\bar{l}) = 0$ and each formula in \mathfrak{W} is false under $\lambda^\#$. Therefore \mathfrak{W} is not valid.

(\leftarrow) Assume \mathfrak{W} is not valid. There exists an extended mbC-valuation $\lambda^\#$, such that each formula in \mathfrak{W} is false under $\lambda^\#$. In this case $\lambda^\#(l) = 1$ and $\lambda^\#(A_1 \vee \dots \vee A_n) = 0$. Therefore $l \not\vdash_{\text{mbC}} A_1 \vee \dots \vee A_n$. ■

LEMMA 12. $l \rightarrow k \not\vdash_{\text{mbC}} A_1 \vee \dots \vee A_n$ (where each A_i ($1 \leq i \leq n$) is a literal) if and only if a dual Hintikka set $\mathfrak{W} = \{l, A_1, \dots, A_n\}$ is not valid or $\mathfrak{W} = \{\bar{k}, A_1, \dots, A_n\}$ is not valid.

PROOF. The proof is analogous to the proof of LEMMA 11. ■

THEOREM 4.5. Each abductive hypothesis generated by Algorithm 1, where each abductive rule is applied with a significance constraint, is significant.

PROOF. The proof is a consequence of LEMMAS 9 and 10 and the construction of $\mathfrak{W}_\Gamma^{nv+}$. ■

THEOREM 4.6. Each abductive hypothesis generated by Algorithm 1, where each abductive rule is applied with a significance and consistency constraint is significant and consistent.

PROOF. The proof follows from THEOREMS 4.5 and 4.4. ■

5. Examples

We shall provide two examples in order to explain how the Algorithm 1 generates abductive hypotheses. At the end of the section some remarks about the way the algorithm works have been provided.

EXAMPLE 1. Let us first consider an abductive question $Q = ?(\Gamma \vdash \Delta)$, where:

- $\Gamma = \langle p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s) \rangle$,
- $\Delta = \langle z \rangle$.

The complete s-transformation of Q assigned to \mathbf{s} is the following:

$$\frac{\frac{\frac{?(p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s)) \vdash z}{?(q, \neg \sim s, p \rightarrow (q \rightarrow r)) \vdash z} \mathbf{L}_\alpha}{?(q, s, \neg \chi \sim s, p \rightarrow (q \rightarrow r)) \vdash z} \mathbf{L}_{\neg \sim}}{\frac{?(\neg p, q, s, \neg \chi \sim s \vdash z ; q \rightarrow r, q, s, \neg \chi \sim s \vdash z) \mathbf{L}_\beta}{?(\neg p, q, s, \neg \chi \sim s \vdash z ; \neg q, q, s, \neg \chi \sim s \vdash z ; r, q, s, \neg \chi \sim s \vdash z) \mathbf{L}_\beta} \mathbf{L}_\beta$$

Open sequents from the last term of \mathbf{s} are assigned to Φ :

- $\langle \neg p, q, s, \neg \chi \sim s \vdash z \rangle$,
- $\langle r, q, s, \neg \chi \sim s \vdash z \rangle$.

The number of elements in Φ : $x = 2$, therefore $\Theta = \{1, 2\}$. The consistency property $\mathfrak{U}_\Gamma^c = \{\mathfrak{U}_\Gamma^1, \mathfrak{U}_\Gamma^2, \mathfrak{U}_\Gamma^3\}$, where:

$$\begin{aligned} \mathfrak{U}_\Gamma^1 &= \{p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s), q, s, \neg \chi \sim s, \neg p, q \rightarrow r, r\}, \\ \mathfrak{U}_\Gamma^2 &= \{p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s), q, s, \neg \chi \sim s, \neg p\}, \\ \mathfrak{U}_\Gamma^3 &= \{p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s), q, s, \neg \chi \sim s, q \rightarrow r, r\}. \end{aligned}$$

The non-validity property contains one set $\mathfrak{W}_\Delta^{nv} = \{\mathfrak{W}_\Delta^1\}$, where:

$$\mathfrak{W}_\Delta^1 = \{z\}.$$

Let us assume that we randomly get $j = 1$. In that case:

$$\phi = \langle \neg p, q, s, \neg \chi \sim s \vdash z \rangle.$$

Let us further assume that we randomly get $r = 1$, therefore $\mathbf{R} = \mathbf{R}_{abd}^1$, and our partial answer generated by means of the rule \mathbf{R} is the following:

$$a = p.$$

In the next step we have to cross out from consistency property those Hintikka sets that are inconsistent with a , i.e. sets \mathfrak{U}_Γ^1 and \mathfrak{U}_Γ^2 . After this step consistency property looks as follows:

$$\mathfrak{U}_\Gamma^c = \{\mathfrak{U}_\Gamma^3\}.$$

Similarly, we leave only those dual Hintikka set in non-validity property that are not d-valid with a . In this case nothing changes, because \mathfrak{W}_Δ^1 is not d-valid with a . The set Ω is enlarged by partial hypothesis $a = p$ and $j = 1$ is removed from Θ , therefore $\Theta = \{2\}$.

Θ still contains one element, which is now assigned to $j = 2$. In this case:

$$\phi = \langle r, q, s, \neg\chi \sim s \vdash z \rangle.$$

At this stage the algorithm can randomly assign $r = 1$ again. However, there is no partial hypothesis that could be generated by means of \mathbf{R}_{abd}^1 in accordance with the consistency constraint for \mathbf{R}_{abd}^1 . The reason is that there is only one Hintikka set in the consistency property, which contains complementary unanalyzable formulas of all formulas that belong to the antecedent of the sequent ϕ . The algorithm cannot execute instructions in the *if* loop and the value for r is again randomly assigned.

Let us assume that $r = 2$ this time. $\mathbf{R} = \mathbf{R}_{abd}^2$ and the generated partial answer can be $a = q \rightarrow z$, since it is consistent with the \mathfrak{U}_Γ^3 and not d-valid with \mathfrak{W}_Δ^1 . The set Ω is enlarged by a and $j = 2$ is removed from Θ , leaving $\Theta = \emptyset$ as a result. Therefore, the condition for breaking the *while* loop is fulfilled. The set Ω is transformed into the abductive hypothesis by linking all partial answers contained in it with conjunction:

$$\Omega = p \wedge (q \rightarrow z).$$

The addition of Ω to the Γ from initial abductive question Q results in obtaining a question $Q^* = ?(p \wedge (q \rightarrow z), p \rightarrow (q \rightarrow r), \neg(q \rightarrow \sim s) \vdash z)$, which is no longer an abductive one.

EXAMPLE 2. Let us now consider an abductive question $Q' = ?(\Gamma' \vdash \Delta')$, where:

- $\Gamma' = \langle (p \vee r) \rightarrow \neg \circ q \rangle,$
- $\Delta' = \langle \sim z \rangle.$

The complete s-transformation of Q' assigned to \mathbf{s} is the following:

$$\frac{\frac{\frac{?((p \vee r) \rightarrow \neg \circ q \vdash \sim z)}{?((p \vee r) \rightarrow \neg \circ q \vdash \neg z, \chi \sim z)} \mathbf{R}_\sim}{\frac{?(\neg(p \vee r) \vdash \neg z, \chi \sim z ; \neg \circ q \vdash \neg z, \chi \sim z)}{?(\neg p, \neg r \vdash \neg z, \chi \sim z ; \neg \circ q \vdash \neg z, \chi \sim z)} \mathbf{L}_{\neg \rightarrow}}{\frac{?(\neg p, \neg r \vdash \neg z, \chi \sim z ; q, \sim q \vdash \neg z, \chi \sim z ; \neg \chi \circ q \vdash \neg z, \chi \sim z)}{\frac{?(\neg p, \neg r \vdash \neg z, \chi \sim z ; q, \neg q \vdash \neg z, \chi \sim z ; q, \chi \sim q \vdash \neg z, \chi \sim z ; \neg \chi \circ q \vdash \neg z, \chi \sim z)}{\mathbf{L}_{\neg \circ}} \mathbf{L}_{\sim}} \mathbf{L}_{\sim}}$$

Open sequents from the last term of \mathbf{s} are assigned to Φ :

- $\langle \neg p, \neg r \vdash \neg z, \chi \sim z \rangle$,
- $\langle q, \chi \sim q \vdash \neg z, \chi \sim z \rangle$,
- $\langle \neg \chi \circ q \vdash \neg z, \chi \sim z \rangle$.

The number of elements in Φ : $x = 3$, therefore $\Theta = \{1, 2, 3\}$. The consistency property $\mathfrak{U}_\Gamma^c = \{\mathfrak{U}_\Gamma^1, \mathfrak{U}_\Gamma^2, \mathfrak{U}_\Gamma^3, \mathfrak{U}_\Gamma^4, \mathfrak{U}_\Gamma^5, \mathfrak{U}_\Gamma^6\}$, where:

$$\begin{aligned} \mathfrak{U}_\Gamma^1 &= \{(p \vee r) \rightarrow \neg \circ q, \neg(p \vee q), \neg \circ q, \neg p, \neg r, q, \sim q, \neg \chi \circ q, \neg q, \chi \sim q\}, \\ \mathfrak{U}_\Gamma^2 &= \{(p \vee r) \rightarrow \neg \circ q, \neg(p \vee q), \neg \circ q, \neg p, \neg r, q, \sim q, \chi \sim q\}, \\ \mathfrak{U}_\Gamma^3 &= \{(p \vee r) \rightarrow \neg \circ q, \neg(p \vee q), \neg \circ q, \neg p, \neg r, \neg \chi \circ q\}, \\ \mathfrak{U}_\Gamma^4 &= \{(p \vee r) \rightarrow \neg \circ q, \neg(p \vee q), \neg p, \neg r\}, \\ \mathfrak{U}_\Gamma^5 &= \{(p \vee r) \rightarrow \neg \circ q, \neg \circ q, q, \sim q, \chi \sim q\}, \\ \mathfrak{U}_\Gamma^6 &= \{(p \vee r) \rightarrow \neg \circ q, \neg \circ q, \neg \chi \circ q\}. \end{aligned}$$

The non-validity property contains one set $\mathfrak{W}_\Delta^{nv} = \{\mathfrak{W}_\Delta^1\}$, where:

$$\mathfrak{W}_\Delta^1 = \{\sim z, \neg z, \chi \sim z\}.$$

Let us assume that we randomly get $j = 1$. In that case:

$$\phi = \langle \neg p, \neg r \vdash \neg z, \chi \sim z \rangle.$$

Let us further assume that we randomly get $r = 2$, therefore $\mathbf{R} = \mathbf{R}_{abd}^2$, and our partial answer generated by means of the rule \mathbf{R} is the following:

$$a = \neg p \rightarrow \neg z.$$

In the next step we have to cross out from consistency property those Hintikka sets that do not fulfil the consistency restriction for \mathbf{R} . After this step consistency property contains the same Hintikka sets:

$$\mathfrak{U}_\Gamma^c = \{\mathfrak{U}_\Gamma^1, \mathfrak{U}_\Gamma^2, \mathfrak{U}_\Gamma^3, \mathfrak{U}_\Gamma^4, \mathfrak{U}_\Gamma^5, \mathfrak{U}_\Gamma^6\}.$$

Similarly, we leave only those dual Hintikka set in non-validity property that are not d-valid with a . In this case nothing changes, because \mathfrak{W}_Δ^1 is not d-valid with a . The set Ω is enlarged by partial hypothesis $a = p \rightarrow \neg z$ and $j = 1$ is removed from Θ , therefore $\Theta = \{2, 3\}$.

Θ still contains elements. Let us assume that we randomly get $j = 2$. In this case:

$$\phi = \langle q, \chi \sim q \vdash \neg z, \chi \sim z \rangle.$$

Let us assume that $r = 5$ this time. $\mathbf{R} = \mathbf{R}_{abd}^5$ and the generated partial answer can be:

$$a = \circ q.$$

The following Hintikka sets do not fulfil the consistency restriction for \mathbf{R} : $\mathfrak{U}_\Gamma^1, \mathfrak{U}_\Gamma^2, \mathfrak{U}_\Gamma^5$, therefore they are removed from the consistency property:

$$\mathfrak{U}_\Gamma^c = \{\mathfrak{U}_\Gamma^3, \mathfrak{U}_\Gamma^4, \mathfrak{U}_\Gamma^6\}.$$

Similarly as in the case of the previous partial hypothesis, a is not d-valid with \mathfrak{W}_Δ^1 and the non-validity property does not change.

The set Ω is enlarged by a and $j = 2$ is removed from Θ , leaving $\Theta = \{3\}$ as a result. Therefore, in the next step $j = 3$ and:

$$\phi = \langle \neg\chi \circ q \vdash \neg z, \chi \sim z \rangle.$$

There are only two abductive hypotheses which can generate a partial hypothesis for ϕ , namely \mathbf{R}_{abd}^1 and \mathbf{R}_{abd}^2 . Assuming that $r = 1$ the partial hypothesis we obtain is of the following form:

$$a = \chi \circ q.$$

Hintikka sets \mathfrak{U}_Γ^3 and \mathfrak{U}_Γ^6 do not fulfil the consistency restriction for \mathbf{R} and are removed from the consistency property:

$$\mathfrak{U}_\Gamma^c = \{\mathfrak{U}_\Gamma^4\}.$$

Since there is a Hintikka set left in the consistency property and a is not d-valid with the \mathfrak{W}_Δ^1 set from the non-validity property, constructed partial hypothesis is significant and valid.

$\Theta = \emptyset$ and the *while* loop is broken. As in the previous example, the set Ω is transformed into the abductive hypothesis by linking all partial answers contained in it with conjunction:

$$\Omega = (\neg p \rightarrow \neg z) \wedge \circ q \wedge \chi \circ q.$$

The question $?(\Gamma' \cup \Omega \vdash \Delta')$ is not an abductive one.

Our algorithm exhibits some weaknesses that should be mentioned. First of all, the algorithm is just a scheme used to depict how the abductive procedure works, rather than optimised implementation of an abductive hypotheses generator. Another point is that the algorithm will not terminate in case when it is impossible to generate partial answer by means of the proposed abductive rules used along with the constraints. There are at least two possible situations of this kind: the abductive goal is inconsistent with

the knowledge base, or already generated partial answers make it impossible to generate further partial answers.

The algorithm is also not optimised for finding the shortest possible abductive hypotheses, nevertheless it is possible that it will find them. It is easy to see that in some cases the algorithm will not recognise, that a partial answer for one open sequent can be a partial answer for other open sequents.

Algorithm 1: Generation of a consistent and valid abductive hypothesis

Input: An abductive question $Q = ?(\Gamma \vdash \Delta)$ of $\mathcal{L}_{\Gamma\text{-mbC}}^2$

Output: An answer Ω to an abductive question Q (abductive hypothesis)

```

1 begin
2    $s \leftarrow$  a complete s-transformation of  $Q$ ;
3    $\Phi \leftarrow$  open sequents from the last term of  $s$ ;
4    $x \leftarrow$  the number of terms in  $\Phi$ ;
5    $\Theta \leftarrow \{1, \dots, x\}$ ;
6    $\mathcal{U}_\Gamma^c \leftarrow \{\mathcal{U}_\Gamma^1, \dots, \mathcal{U}_\Gamma^n\}$  the consistency property based on  $\Gamma$ ;
7    $\mathcal{W}_\Delta^{nv} \leftarrow \{\mathcal{W}_\Delta^1, \dots, \mathcal{W}_\Delta^m\}$  the non-validity property based on  $\Delta$ ;
8    $\Omega \leftarrow \emptyset$ ;
9   while  $\Theta$  non-empty do
10     $j \leftarrow$  random element from  $\Theta$ ;
11     $\phi \leftarrow$  the  $j$ -th term of  $\Phi$ ;
12     $r \leftarrow$  random element from the set  $\{1, 2, 3, 4, 5\}$ ;
13     $\mathbf{R} \leftarrow$  abductive rule  $\mathbf{R}_{abd}^r$  ( $\mathbf{R}_{abd}^{r*}$ );
14    if  $\mathbf{R}$  can be applied to  $\phi$  with constraint for  $\mathbf{R}$  then
15       $a \leftarrow$  partial answer from  $\mathbf{R}$ ;
16       $\mathcal{U}_\Gamma^c \leftarrow \mathcal{U}_\Gamma^c \setminus \{\mathcal{U}_\Gamma^i \in \{\mathcal{U}_\Gamma^1, \dots, \mathcal{U}_\Gamma^n\} \mid \mathcal{U}_\Gamma^i \text{ is not compatible under}$ 
the consistency constraint for  $\mathbf{R}$  with  $a$  as a partial answer}\};
17       $\mathcal{W}_\Delta^{nv} \leftarrow \mathcal{W}_\Delta^{nv} \setminus \{\mathcal{W}_\Delta^i \in \{\mathcal{W}_\Delta^1, \dots, \mathcal{W}_\Delta^m\} \mid \mathcal{W}_\Delta^i \text{ is not}$ 
compatible under the validity constraint for  $\mathbf{R}$  with  $a$  as a
partial answer}\};
18       $\Omega \leftarrow \Omega \cup \{a\}$ ;
19       $\Theta \leftarrow \Theta \setminus \{j\}$ ;
20    end
21  end
22   $\Omega \leftarrow \bigwedge \Omega$ ;
23 end

```

6. Discussion

In this section we want to compare our model of abductive reasoning with two other approaches.

6.1. Carnielli's System

There was an earlier attempt, made by Carnielli, to model abductive reasoning in the context of paraconsistent logic. We will briefly compare his system to ours (for details, see [3]).

As we have mentioned at the beginning of the paper, four ingredients of algorithmic approach to abduction can be distinguished: a basic logic (which gives us a formal language and a system of formulas considered *valid*), proof method (a way, in which this logic is given), hypotheses generation mechanism and criteria which rule out hypotheses which are not good enough. The logic used by Carnielli is LFI1 system, which is an extension of mbC for which a simple 3-valued semantics exists. As a proof method, signed version of analytic tableaux is used. Passing to the last two ingredients, the situation becomes more complicated: it seems to us that the procedure of generating hypotheses and hypotheses evaluation mechanism are interrelated. Such properties of hypotheses as consistency (non-triviality in Carnielli system), analyticity or minimality are forced by the very definition of what abductive problem and abductive solution are (see definition 5.1 in [3]). It is an approach very different from ours: we try to be neutral in the definition of hypotheses generation mechanism with respect to properties one may think of as desirable. But we also deliver some simple implementation of most frequently accepted properties such as consistency and significance, but we let the users to decide, which properties they want to deploy. Another major difference between these systems is the form of hypotheses, which can be generated. In Carnielli's system, hypotheses are collections of atoms. This is determined by the proof method used. One of the consequences of this approach is that disjunctive hypotheses are impossible to obtain (we will say more on this, when discussing some specific examples). Another is that only analytic hypotheses are accepted. Our approach is more general: we stipulate in what way an abductive rule, which enables hypotheses generation, has to function and we provide two examples, one of them enabling *law-like* hypotheses (see [25]). This approach is, in a way, open ended: new abductive rules can be added, which enable more interesting hypotheses, in particular non-analytic ones, which is impossible in Carnielli's system.

Let us consider an eminently simple abductive problem: we want to obtain q from the knowledge base consisting solely of p . In the context of AQAS one can pose an abductive question

$$?(p \vdash q)$$

In Carnielli's system we start by constructing a tableaux, where we list all formulas from the knowledge base (and we assign to each of them symbol **T**) and an abductive goal (with **F** assigned) in the root of a tree.

$$\begin{array}{c} \mathbf{T}p \\ | \\ \mathbf{F}q \end{array}$$

There is no rule which can be applied to simplify our problem further. In Carnielli's system two abductive hypothesis can be generated: **F** p and **T** q . The first one is (classically) inconsistent with the knowledge base. The second one is certainly too strong: from **T** q follows that q is true, but that is exactly what we were trying to explain.

In AQAS we start with the aforementioned question

$$?(p \vdash q)$$

which cannot be further simplified.

By means of question-answer rules we can formulate three hypotheses: $\neg p$ (where \neg denotes classical negation), q and $p \rightarrow q$. The first two hypotheses do not meet consistency and significance restriction, but the third one does.

This simple example shows that AQAS is able to generate good (in the above case *the only good*) hypotheses which are not reachable for Carnielli's system.

In the next example we will show that AQAS produces more good hypotheses than other considered systems. Our knowledge base consists of $p \rightarrow q$ and $q \rightarrow r$, and we want to derive r . The initial step is the transformation of an abductive question:

$$\frac{\frac{\frac{?(p \rightarrow q, q \rightarrow r \vdash r)}{?(\neg p, q \rightarrow r \vdash r ; q, q \rightarrow r \vdash r)}}{?(\neg p, \neg q \vdash r ; \neg p, r \vdash r ; q, q \rightarrow r \vdash r)}}{?(\neg p, \neg q \vdash r ; \neg p, r \vdash r ; q, \neg q \vdash r ; q, r \vdash r)}$$

There is only one open constituent of the last question, namely the sequent $\neg p, \neg q \vdash r$ and it can be closed in AQAS (with consistency and significance constraints) by any of the following formulas: $p, q, \neg p \rightarrow r, \neg q \rightarrow r$. In Carnielli's system only p and q can be obtained due to the fact that abductive hypothesis is considered there as a conjunction of literals, thus more

complicated formulas are not reachable, which can result in a system not producing any hypothesis despite the fact that there is one, as the previous example has shown.

6.2. Abductive Logic Programming

The next abductive procedure that we want to compare our approach with is Abductive Logic Programming (ALP) (details of the method can be found in [16]). The ALP framework consists of three ingredients: a logic program \mathcal{P} (knowledge base), a set of abducibles A (i.e. potential abductive hypotheses) and a set of integrity constraints IC , where we can express constraints regarding our knowledge that are additional to the logic program \mathcal{P} . ALP is aimed at modeling the syllogistic perspective on abductive reasoning [12].

Logic used in ALP is a part of the first order logic called logic programming where only universally quantified implications are used. The antecedent of the implication is a set of literals, and the consequent of the implication is an atom. In ALP all variables are substituted in a consequent manner by constants from a finite set, therefore we use propositional examples for ALP. The proof method is the standard SLD-resolution with backward reasoning in SLD-fashion. The mechanisms for the generation and evaluation of abductive hypotheses are mixed together. One of the starting ingredients of ALP is a set of abducibles A , which is assumed to be given at the start. In [16] set A contains all atoms that occur in logic program \mathcal{P} or atoms that occur in logic program \mathcal{P} but only those that do not appear in the consequent of any implication. Abductive hypotheses are defined as subsets of A set. Therefore, the latter method for A set generation impose the minimality restriction for abductive hypotheses. In addition, the proof procedure accompanied by the above mentioned method for the set of abducibles generation guarantees that obtained abductive hypotheses are consistent with the initial knowledge base represented by the the logic program \mathcal{P} and the set of integrity constraints IC .

Differences between the approach described in this paper and the ALP occur in the foundations of both methods. In ALP abductive goals and hypotheses are restricted only to literals, while we do not have such restriction regarding abductive goals in our procedure and generated abductive hypotheses include also law-like statements. Furthermore, we generate abductive hypotheses, while in ALP framework abductive hypotheses are picked from the set of abducibles given from the start. As a consequence, we are able to produce abductive hypotheses for abductive goals that contain information that do not belong to the initial knowledge base, contrary to

the ALP approach. There are many implementations of Abductive Logic Programming, e.g. in Prolog [22] or in a neuro-symbolic system [14].

Let us consider the same two examples as in the previous subsection. For the first abductive problem, where we have p as our knowledge base and q as the abductive goal, our procedure is able to generate three abductive hypotheses, i.e. $\neg p$, q and $p \rightarrow q$, with one meeting both, consistency and significance restrictions. In this case the ALP procedure is not able to produce any hypothesis.

In the second example we have the following formulas as the knowledge base:

$$p \rightarrow q, q \rightarrow r$$

and as the abductive goal we have r . Our approach generates the following four abductive hypotheses that are consistent and significant: p , q , $\neg p \rightarrow q$, $\neg q \rightarrow r$. For the ALP procedure we assume that $\mathcal{P} = \{q \leftarrow p, r \leftarrow q\}$ and the set of integrity constraints IC is empty. According to the two ways of selecting the set of abducibles we have the following: $A = \{p, q\}$ or $A' = \{p\}$. In the first case ALP produces the following three hypotheses: $\{p\}$, $\{q\}$ and $\{p, q\}$. In the second case we have only one abductive hypothesis $\{p\}$.

7. Summary and Conclusion

In this paper we have introduced an Abductive Question-Answer System for the minimal logic of formal inconsistency **mbC**. The system produces abductive hypotheses, which are answers to abductive questions concerning derivability of formulas from sets of formulas. We integrated generation and evaluation of hypotheses via constraints of consistency and significance being imposed on the system rules. Our further research will be concerned with optimization issues. We also plan for modular implementation of more diverse set of evaluation criteria, which would allow for producing hypotheses exhibiting different characteristics, depending on particular choice of criteria.

Additionally, we have compared our procedure with two alternative approaches. Generally speaking, abductive hypotheses in ALP and Carnielli's system are conjunction of literals, where in AQAS each hypothesis can be considered as a conjunctive normal form, which can consist of nonempty disjunctions (due to the interdefinability of implication and disjunction in the presence of classical negation). In both cases the model of abductive

reasoning proposed in this paper is more flexible with regards to assuming abductive goals and creation of abductive hypotheses. In addition, our system clearly distinguishes between the generation and evaluation of abductive hypotheses, while in both other approaches this division is not plain and simple.

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References

- [1] ALISEDA, A., *Abductive reasoning: Logical investigations into discovery and explanation*, vol. 330 of *Synthese Library*, Springer Science & Business Media, 2006.
- [2] BATENS, D., Paraconsistent extensional propositional logics, *Logique et Analyse* 23(90–91):195–234, 1980.
- [3] CARNIELLI, W., Surviving abduction, *Logic Journal of IGPL* 14(2):237–256, 2006.
- [4] CARNIELLI, W., and J. MARCOS, Tableau systems for logics of formal inconsistency, in *Proceedings of the 2001 International Conference on Artificial Intelligence*, vol. 2, 2001, pp. 848–852.
- [5] CARNIELLI, W., and J. MARCOS, A taxonomy of c-systems, *arXiv preprint math/0108036*, 2001.
- [6] CARNIELLI, W.A., M. E. CONIGLIO, and J. MARCOS, Logics of formal, in F. Guenther, and D. M. Gabbay, (eds.), *Handbook of Philosophical Logic*, vol. 14, Springer, 2013, pp. 1–93.
- [7] CHLEBOWSKI, SZ., *Canonical and Dual Erotetic Calculi for First-Order Logic*, Ph.D. thesis, Adam Mickiewicz University in Poznań, 2018.
- [8] CHLEBOWSKI, SZ., and D. LESZCZYŃSKA-JASION, Dual Erotetic Calculi and the Minimal LFI, *Studia Logica* 103(6):1245–1278, 2015.

- [9] CHLEBOWSKI, SZ., and D. LESZCZYŃSKA-JASION, *Erotetic calculi, duality and proof-search*, tech. rep., Chair of Logic and Cognitive Science, Adam Mickiewicz University, 2015.
- [10] CONIGLIO, M.E., and T.G. RODRIGUES, Some investigations on mbC and mCi, in C.A. Mortari, (ed.), *Tópicos de lógicas não clássicas*, NEL/UFSC, 2014, pp. 11–70.
- [11] FITTING, M., *First-order logic and automated theorem proving*, Springer Science & Business Media, 2012.
- [12] FLACH, P. A., and A. C. KAKAS, *Abduction and Induction: Essays on their relation and integration*, vol. 18 of *Applied Logic Series*, Springer Science & Business Media, 2013.
- [13] GABBAY, D.M., and J. WOODS, *The Reach of Abduction. Insight and Trial*, Elsevier, 2005.
- [14] GARCEZ, A., D. M GABBAY, O. RAY, and J. WOODS, Abductive reasoning in neural-symbolic systems, *Topoi* 26(1):37–49, 2007.
- [15] HINTIKKA, J., *Inquiry as Inquiry: A Logic of Scientific Discovery*, Kluwer, Dordrecht/Boston/London, 1999.
- [16] KAKAS, A. C., R. A. KOWALSKI, and F. TONI, Abductive logic programming, *Journal of logic and computation* 2(6):719–770, 1992.
- [17] KOMOSINSKI, M., A. KUPS, D. LESZCZYŃSKA-JASION, and M. URBAŃSKI, Identifying efficient abductive hypotheses using multicriteria dominance relation, *ACM Transactions on Computational Logic* 15(4):28:1–28:20, 2014.
- [18] LESZCZYŃSKA-JASION, D., *The Method of Socratic Proofs for Normal Modal Propositional Logics*, Adam Mickiewicz University Press, Poznań, 2007.
- [19] LOPARIC, A., Semantical study of some propositional calculi, *Journal of Symbolic Logic* 43:358, 1978.
- [20] NETO, A.G.S.S., and M. FINGER, Effective prover for minimal inconsistency logic, in M. Bramer, (ed.), *IFIP International Federation for Information Processing*, Springer, 2006, pp. 465–474.
- [21] PEIRCE, C. S., *Collected papers of charles sanders peirce*, vol. 5, Harvard University Press, 1974.
- [22] SAPTAWIJAYA, A., and L. M. PEREIRA, Tabdual: a tabled abduction system for logic programs, *FLAP* 2(1):69–124, 2015.
- [23] THAGARD, P., Abductive inference: From philosophical analysis to neural mechanisms, in A. Feeney, and E. Heit, (eds.), *Inductive reasoning: Experimental, developmental, and computational approaches*, Cambridge University Press, 2007, pp. 226–247.
- [24] URBAŃSKI, M., and A. KLAWITER, Abduction: Some conceptual issues, *Logic and Logical Philosophy* 27(4):583–597, 2018.
- [25] URBAŃSKI, M., and A. WIŚNIEWSKI, On search for law-like statements as abductive hypotheses by socratic transformations, in C. Baskent, (ed.), *Perspectives on Interrogative Models of Inquiry. Developments in Inquiry and Questions*, Springer, 2016, pp. 111–127.
- [26] WIŚNIEWSKI, A., *The Posing of Questions: Logical Foundations of Erotetic Inferences*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1995.
- [27] WIŚNIEWSKI, A., Socratic Proofs, *Journal of Philosophical Logic* 33(3):299–326, 2004.

- [28] WIŚNIEWSKI, A., *Questions, Inferences and Scenarios*, vol. 46 of *Studies in Logic. Logic and Cognitive Systems*, College Publications, London, 2013.
- [29] WIŚNIEWSKI, A., and V. SHANGIN, Socratic Proofs for Quantifiers, *Journal of Philosophical Logic* 35(2):147–178, 2006.

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